ON HIGHER ORDER BOURGAIN ALGEBRAS OF A NEST ALGEBRA

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Abstract. Following earlier work in which we provided algebraic characterizations of the right, left, and two-sided Bourgain algebras, as well as the second order Bourgain algebras, associated with a nest algebra, we herein demonstrate that a given nest algebra has (essentially) at most six different third order Bourgain algebras, and that every fourth order (or higher) Bourgain algebra of the nest algebra coincides with one of at most third order. This puts the final touch on the description of Bourgain algebras of nest algebras.

Following the work of Bourgain in [1], the notion of what is now referred to as the 'Bourgain algebra' associated with a subset X of a $C(K)$ space was formalized by Cima, et al ([2], [3]), who, among numerous other things, showed it to be a norm-closed algebra. The Bourgain algebras of various function spaces have since been studied by several authors (cf. [7], [8]). In [5] and [6], working in the non-commutative setting of operators, we formulated analogous definitions, in topological terms, of right, left, and two-sided Bourgain algebras associated to a given algebra of operators, and provided algebraic characterizations of these in the case where the given operator algebra is a nest algebra. The second order Bourgain algebras associated to a nest algebra were characterized as well. In this note, we demonstrate that a given nest algebra has (essentially) at most six possible third order Bourgain algebras, and that every fourth order (or higher) Bourgain algebra of the nest algebra coincides with one of at most third order. This puts the final touch on the description of the Bourgain algebras associated with nest algebras.

1. Preliminaries.

In this note, $\mathcal{H}$ will denote a separable, infinite-dimensional Hilbert space while $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ will denote the algebra of bounded linear operators on $\mathcal{H}$ and the ideal of compact operators in $\mathcal{L}(\mathcal{H})$, respectively. All projections in $\mathcal{L}(\mathcal{H})$ are assumed to be self-adjoint and all subspaces closed. For P a projection in $\mathcal{L}(\mathcal{H})$, we denote the orthogonal complement by $P^{\perp} = (1-P)$.

A *nest* is a set of projections in $\mathcal{L}(\mathcal{H})$ which is totally ordered by range inclusion, contains 0 and 1, and is closed under suprema and infima. The

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nest algebra, alg$\mathcal{N}$, corresponding to a given nest $\mathcal{N}$, is the algebra of operators leaving invariant the ranges of all projections in $\mathcal{N}$; that is, alg$\mathcal{N} = \{T \in \mathcal{L}(\mathcal{H}) : PTP = TP \text{ for all } P \in \mathcal{N}\}$. For a wealth of information on nest algebras, refer to [4].

For a nest $\mathcal{N}$ and a projection P in $\mathcal{N}$, define the projections $P_- = \sup\{E \in \mathcal{N} : E < P\}$ and $P_+ = \inf\{E \in \mathcal{N} : E > P\}$. As the nest is complete (closed under suprema and infima), it follows that P_- and P_+ are both in $\mathcal{N}$. An *interval* in $\mathcal{N}$ is a projection of the form $(F - E)$, where E and F are in $\mathcal{N}$ and $E < F$. A minimal interval in a nest is called an *atom*.

In this note, we will make use of the following three subnests of a given nest $\mathcal{N}$. Define

$$\begin{aligned} \mathcal{N}_r &= \mathcal{N}\backslash\{P \in \mathcal{N} : P_- < P < P_+ \text{ and } \dim(P - P_-) < \infty\}, \\ \mathcal{N}_l &= \mathcal{N}\backslash\{P \in \mathcal{N} : P_- < P < P_+ \text{ and } \dim(P_+ - P) < \infty\}, \\ \mathcal{N}_\infty &= \mathcal{N}\backslash\{P \in \mathcal{N} : P_- < P < P_+ \text{ and } \dim(P_+ - P_-) < \infty\}. \end{aligned}$$

Observe that each of these is itself a complete nest as no strong limit P of $\mathcal{N}$ can satisfy $P_- < P < P_+$. With $\mathcal{A} = \text{alg}\mathcal{N}$, let $\mathcal{A}_r$, $\mathcal{A}_l$, and $\mathcal{A}_\infty$ be the nest algebras corresponding to these subnests, respectively. It is evident that $\mathcal{N}_r \cup \mathcal{N}_l = \mathcal{N}_\infty \subseteq \mathcal{N}$ and, therefore, that $\mathcal{A}_r \cap \mathcal{A}_l = \mathcal{A}_\infty \supseteq \mathcal{A}$.

By analogy with the function-theory case, but in the non-commutative operator setting, we associate to each subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ three Bourgain-type algebras. First, the *right Bourgain algebra* of $\mathcal{A}$ is defined by

$$\mathcal{A}_b = \{T \in \mathcal{L}(\mathcal{H}) : \text{dist}(TA_n, \mathcal{A}) \to 0 \text{ whenever } A_n \to 0 \text{ weak}-^* \text{ in } \mathcal{A}\}.$$

(Recall that a sequence $\{A_n\}$ in $\mathcal{L}(\mathcal{H})$ converges to 0 in the weak-* topology if, and only if, the traces $\text{tr}(XA_n)$ converge to 0 for every fixed trace-class operator X.) Similarly, define the *left Bourgain algebra* of $\mathcal{A}$ by

$${}_b\mathcal{A} = \{T \in \mathcal{L}(\mathcal{H}) : \text{dist}(A_nT, \mathcal{A}) \to 0 \text{ whenever } A_n \to 0 \text{ weak}-^* \text{ in } \mathcal{A}\}.$$

Finally, the *two-sided Bourgain algebra*, or, simply, the *Bourgain algebra* of $\mathcal{A}$ is given by

$$\mathcal{A}_B = \mathcal{A}_b \cap {}_b\mathcal{A}.$$

The demonstration that each of these Bourgain algebras is, in fact, a norm-closed subalgebra of $\mathcal{L}(\mathcal{H})$ containing the algebra $\mathcal{A}$ is left as an exercise.

For a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$, we can define higher order (two-sided) Bourgain algebras of $\mathcal{A}$ by

$$(\mathcal{A})_{B^{(n)}} \equiv ((\mathcal{A}_B)\dots)_B\,,\ (n \text{ times}),$$

for $n \geq 1$. In like manner, n^{th} order right and left Bourgain algebras, $(\mathcal{A})_{b^{(n)}}$ and ${}_{b^{(n)}}(\mathcal{A})$, respectively, can also be defined as can higher order 'mixed' Bourgain algebras. Naturally, every n^{th} order (two-sided, right, left, or mixed) Bourgain algebra of $\mathcal{A}$ is a norm-closed algebra containing $\mathcal{A}$ itself.

2. Higher order Bourgain algebras of a nest algebra.

In [6], we proved the following two Theorems giving algebraic characterizations of the three Bourgain algebras and the nine possible second order Bourgain algebras of a nest algebra.

Theorem A. *([6, Theorem 2.4].) Let* $\mathcal{A} = \mathrm{alg}\mathcal{N}$ *be a nest algebra. Then*

(i) $\mathcal{A}_b = (\mathcal{A} + \mathcal{K}) \cap \mathcal{A}_r$,
(ii) ${}_b\mathcal{A} = (\mathcal{A} + \mathcal{K}) \cap \mathcal{A}_l$, *and*
(iii) $\mathcal{A}_B = (\mathcal{A} + \mathcal{K}) \cap \mathcal{A}_\infty$.

Theorem B. *([6, Theorem 3.1].) Let* $\mathcal{N}$ *be a nest with corresponding nest algebra* $\mathcal{A} = \mathrm{alg}\mathcal{N}$. *Then the following statements hold.*

(i) $(\mathcal{A}_B)_b = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_\infty)_r$;
(ii) ${}_b(\mathcal{A}_B) = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_\infty)_l$;
(iii) $(\mathcal{A}_B)_B = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_\infty)_\infty$;
(iv) $(\mathcal{A}_b)_b = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_r)_r$;
(v) ${}_b(\mathcal{A}_b) = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_r)_l$;
(vi) $(\mathcal{A}_b)_B = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_r)_\infty$;
(vii) ${}_b({}_b\mathcal{A}) = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_l)_l$;
(viii) $({}_b\mathcal{A})_b = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_l)_r$;
(ix) $({}_b\mathcal{A})_B = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}_l)_\infty$.

Suppose, now, that $\mathcal{B}$ denotes some n^{th} order Bourgain algebra of the nest algebra $\mathcal{A} = \mathrm{alg}\mathcal{N}$. Thus, $\mathcal{B}$ is obtained from $\mathcal{A}$ by a string of n steps, each one a right, left, or two-sided application. Let $\widetilde{\mathcal{N}}$ be the subnest of $\mathcal{N}$ obtained by applying to $\mathcal{N}$ the corresponding string of r's, l's, and ∞'s. (For instance, if $\mathcal{B} = {}_b((\mathcal{A}_b)_B)$, then $\widetilde{\mathcal{N}} = ((\mathcal{N}_r)_\infty)_l$.) Let $\widetilde{\mathcal{A}} = \mathrm{alg}\widetilde{\mathcal{N}}$. A reasonably straighforward adaptation of the proof of Theorem B (which we leave to the interested reader) then gives the following characterization of all higher order Bourgain algebras of a nest algebra.

Theorem 2.1. *Let* $\mathcal{N}$ *be a nest and* $\mathcal{A} = \mathrm{alg}\mathcal{N}$. *Let* $\mathcal{B}$, $\widetilde{\mathcal{N}}$, *and* $\widetilde{\mathcal{A}}$ *be as just described. If* $\mathcal{B} = \widetilde{\mathcal{A}} \cap (\mathcal{A} + \mathcal{K})$, *then*

(i) $\mathcal{B}_b = (\widetilde{\mathcal{A}})_r \cap (\mathcal{A} + \mathcal{K})$,
(ii) ${}_b\mathcal{B} = (\widetilde{\mathcal{A}})_l \cap (\mathcal{A} + \mathcal{K})$, *and*
(iii) $\mathcal{B}_B = (\widetilde{\mathcal{A}})_\infty \cap (\mathcal{A} + \mathcal{K})$.

Theorem A guarantees that the hypothesis of Theorem 2.1 holds for first order Bourgain algebras. Hence, by induction, the characterizations of Theorem 2.1 are valid for Bourgain algebras of all orders. It follows from this that, to distinguish the various Bourgain algebras of a given nest algebra, it suffices to distinguish the various subnests $\widetilde{\mathcal{N}}$ that can arise, to which task we now turn.

A first observation, used in the proof of Theorem B, is that the nest $\mathcal{N}_\star$, where $\star$ is any of r, l, or ∞, cannot contain two adjacent finite-dimensional atoms $(P - E)$ and $(F - P)$ as this would imply both $P_- < P < P_+$ and $\dim(P_+ - P_-) < \infty$ in $\mathcal{N}$, thereby precluding P from membership in $\mathcal{N}_\star$.

A corollary to this observation is that $(\mathcal{N}_\star)_\infty = \mathcal{N}_\star$, where again $\star$ is any of r, l, or ∞. In other words, from the second order Bourgain algebra on, forming a two-sided Bourgain algebra changes nothing. (Together with Theorem 2.1 above, this generalizes [6, Corollary 3. 2] which showed that $(\mathcal{A}_B)_B = \mathcal{A}_B$ for every nest algebra $\mathcal{A}$.)

Second, note that, if $(P - E)$ is an atom of the subnest $(\mathcal{N}_\star)_\sharp$, where $\star$ and $\sharp$ are any of r, l, or ∞, then it must be the case that $P > P_-$ in the subnest $\mathcal{N}_\star$. Indeed, if $(P - E)$ is an atom in $(\mathcal{N}_\star)_\sharp$ but $P = P_-$ in $\mathcal{N}_\star$, then $\mathcal{N}_\star$ must contain an increasing sequence of projections $\{Q_n : n \geq 1\}$ such that $\dim(Q_{n+1} - Q_n) < \infty$ for all $n \geq 1$. However, by its very definition, $\mathcal{N}_\star$ cannot contain such projections. Similarly, if $(\mathcal{N}_\star)_\sharp$ contains the atom $(F - P)$, then we must have $P < P_+$ in $\mathcal{N}_\star$.

We now deduce that the subnest $(\mathcal{N}_\star)_r$ cannot contain two consecutive atoms of which the first is finite-dimensional. For suppose that $(\mathcal{N}_\star)_r$ contains the atoms $(P - E)$ and $(F - P)$, with $\dim(P - E) < \infty$. By the preceding observation, it must be the case that $P < P_+$ in $\mathcal{N}_\star$. It now follows that $P_- < P < P_+$ in $\mathcal{N}_\star$ with $\dim(P - P_-) < \infty$, which, however, implies that P is not in $(\mathcal{N}_\star)_r$ at all. A similar argument shows that the subnest $(\mathcal{N}_\star)_l$ cannot contain two consecutive atoms of which the second is finite-dimensional.

Another consequence of this last observation is that any maximal string of two or more consecutive atoms in the subnest $(\mathcal{N}_\star)_\sharp$ must have resulted from a maximal string of atoms in $\mathcal{N}_\star$ *having the same endpoints*.

From these observations, we see that, given any maximal string of two or more consecutive atoms in $\mathcal{N}_\star$, passing to the subnest $(\mathcal{N}_\star)_r$ will result in a maximal string of consecutive atoms all of which are infinite-dimensional except possibly one occurring at the upper end of the string (if there is such an atom at all). Similarly, passing to the subnest $(\mathcal{N}_\star)_l$ yields a maximal string of consecutive atoms of which only the atom at the lower end of the string (if such an atom is present) can be finite-dimensional. It follows from this that any maximal string of two or more consecutive atoms in $(\mathcal{N}_\star)_r$ will be preserved in passing to $((\mathcal{N}_\star)_r)_r$. Thus, $(\mathcal{N}_\star)_r = ((\mathcal{N}_\star)_r)_r$, and, similarly, $(\mathcal{N}_\star)_l = ((\mathcal{N}_\star)_l)_l$. (Together with Theorem 2.1 above, this generalizes [6, Corollary 3. 3] which asserts that $(\mathcal{A})_{b^{(3)}} = (\mathcal{A})_{b^{(2)}}$ and ${}_{b^{(3)}}(\mathcal{A}) = {}_{b^{(2)}}(\mathcal{A})$.)

Our final observation is that, as any maximal string of two or more consecutive atoms in the subnest $((\mathcal{N}_\star)_r)_l$ must have resulted from a maximal string of atoms in $(\mathcal{N}_\star)_r$, and, as every atom in this string in $(\mathcal{N}_\star)_r$ is infinite-dimensional except possibly the uppermost, then every atom in the maximal string in $((\mathcal{N}_\star)_r)_l$ must be infinite-dimensional. The same statement holds for $((\mathcal{N}_\star)_l)_r$. It follows that the two third order subnests $((\mathcal{N}_\star)_r)_l$ and $((\mathcal{N}_\star)_l)_r$ are stable.

These observations and results are summarized in the following Proposition.

Proposition 2.2. *Let $\mathcal{N}$ be a nest. Then the following statements hold.*

(i) *$\mathcal{N}_\star$ cannot contain two consecutive finite-dimensional atoms, where $\star$ denotes any of r, l, or ∞.*
(ii) *$(\mathcal{N}_\star)_\infty = \mathcal{N}_\star$, where $\star$ is any of r, l, or ∞.*
(iii) *$(\mathcal{N}_\star)_r$ cannot contain two consecutive atoms of which the first is finite-dimensional. $(\mathcal{N}_\star)_l$ cannot contain two consecutive atoms of which the second is finite-dimensional.*
(iv) *$(\mathcal{N}_\star)_r = ((\mathcal{N}_\star)_r)_r$ and $(\mathcal{N}_\star)_l = ((\mathcal{N}_\star)_l)_l$.*
(v) *$((\mathcal{N}_\star)_r)_l$ and $((\mathcal{N}_\star)_l)_r$ are stable; that is, $(((\mathcal{N}_\star)_r)_l)_\sharp = ((\mathcal{N}_\star)_r)_l$ and $(((\mathcal{N}_\star)_l)_r)_\sharp = ((\mathcal{N}_\star)_l)_r$, where $\star$ and $\sharp$ are any of r, l, or ∞.*

Taken together, Theorem 2.1 and Proposition 2.2 imply that, for a given nest algebra $\mathcal{A} = \text{alg}\mathcal{N}$, every Bourgain algebra must coincide with one of those listed in Theorem B or one of the third order algebras in the set

$$\{{}_b((\mathcal{A}_b)_b),\ {}_b((\mathcal{A}_B)_b),\ {}_b(({}_b\mathcal{A})_b),\ ({}_b(\mathcal{A}_b))_b,\ ({}_b(\mathcal{A}_B))_b,\ ({}_b({}_b\mathcal{A}))_b\}.$$

In other words, $\mathcal{A}$ can have at most fifteen distinct Bourgain algebras of all orders combined (three distinct first order algebras, six additional second order algebras, and a further six of the third order).

Example 2.3. We conclude with an example of a nest algebra for which the fifteen Bourgain algebras just listed are all different and yet which is reasonably manageable. The nest for this algebra consists of three pieces put together.

First consider the nest $\mathcal{N}_1$ of projections on the Hilbert space $\mathcal{H}_1$ defined like so.

$$\mathcal{N}_1 = \{0,\, 1\} \cup \{Q\} \cup \{E_n \,:\, n \in \mathbb{Z}\} \cup \{P\},$$

where the projections $\{E_n\}$ decrease strongly to Q as $n \to -\infty$, where $\{E_n\}$ increases strongly to P as $n \to \infty$, where Q and $P^\perp$ are both finite-dimensional, where $(E_{n+1} - E_n)$ is finite-dimensional for all integers $n \neq 0$, and where $(E_1 - E_0)$ is infinite-dimensional.

Careful consideration shows that, in this example,

$$\begin{aligned}
(\mathcal{N}_1)_r &= ((\mathcal{N}_1)_r)_\infty = \{0,\, Q,\, E_1,\, P,\, 1\}, \\
(\mathcal{N}_1)_l &= ((\mathcal{N}_1)_l)_\infty = \{0,\, Q,\, E_0,\, P,\, 1\}, \text{and} \\
(\mathcal{N}_1)_\infty &= ((\mathcal{N}_1)_\infty)_\infty = \{0,\, Q,\, E_0,\, E_1,\, P,\, 1\}, \\
((\mathcal{N}_1)_r)_r &= \{0,\, E_1,\, P,\, 1\}, \\
((\mathcal{N}_1)_l)_r &= \{0,\, E_0,\, P,\, 1\}, \\
((\mathcal{N}_1)_\infty)_r &= \{0,\, E_0,\, E_1,\, P,\, 1\}, \\
((\mathcal{N}_1)_r)_l &= \{0,\, Q,\, E_1,\, 1\}, \\
((\mathcal{N}_1)_l)_l &= \{0,\, Q,\, E_0,\, 1\}, \text{ and} \\
((\mathcal{N}_1)_\infty)_l &= \{0,\, Q,\, E_0,\, E_1,\, 1\}.
\end{aligned}$$

Thus, all nine first and second order Bourgain algebras are distinct for the nest algebra alg$\mathcal{N}_1$. (This example is substantially simpler than [6, Example 3. 4].)

Next, take $\mathcal{N}_2$ to be the nest of projections on the Hilbert space $\mathcal{H}_2$ given by

$$\begin{aligned}\mathcal{N}_2 &= \{0,\, \mathcal{V}_1,\, E_1,\, E_2,\, \{P_n \,:\, n \geq 1\}, \\ &\qquad E_3,\, E_4,\, \{Q_n \,:\, n \geq 1\},\, E_5,\, E_6,\, E_6 + \mathcal{V}_2,\, 1\},\end{aligned}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are continuous nests on the Hilbert spaces $E_1\mathcal{H}_2$ and ${E_6}^{\perp}\mathcal{H}_2$, respectively, where $(E_2 - E_1)$, $(E_4 - E_3)$, and $(E_6 - E_5)$ are finite-dimensional projections, where the sequence $\{P_n\}$ decreases strongly to E_2 and $\dim(E_3 - P_1) = \infty$, and where the sequence $\{Q_n\}$ decreases strongly to E_4 and $(E_5 - Q_1)$ is infinite-dimensional.

Thirdly, let $\mathcal{N}_3$ be the nest on the Hilbert space $\mathcal{H}_3$ defined by

$$\begin{aligned}\mathcal{N}_3 &= \{0,\, \mathcal{W}_1,\, F_1,\, F_2,\, \{R_n \,:\, n \geq 1\}, \\ &\qquad F_3,\, F_4,\, \{S_n \,:\, n \geq 1\},\, F_5,\, F_6,\, F_6 + \mathcal{W}_2,\, 1\},\end{aligned}$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are continuous nests on the Hilbert spaces $F_1\mathcal{H}_3$ and ${F_6}^{\perp}\mathcal{H}_3$, respectively, where $(F_2 - F_1)$, $(F_4 - F_3)$, and $(F_6 - F_5)$ are finite-dimensional projections, where the sequence of projections $\{R_n\}$ increases strongly to F_3 and $\dim(R_1 - F_2) = \infty$, and where the projections $\{S_n\}$ increase strongly to F_5 and $(S_1 - F_4)$ is infinite-dimensional.

Now put these pieces together by forming the nest

$$\mathcal{M} = \mathcal{N}_1 \oplus 0_{\mathcal{H}_2} \oplus 0_{\mathcal{H}_3} \;\cup\; 1_{\mathcal{H}_1} \oplus \mathcal{N}_2 \oplus 0_{\mathcal{H}_3} \;\cup\; 1_{\mathcal{H}_1} \oplus 1_{\mathcal{H}_2} \oplus \mathcal{N}_3$$

of projections on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. The nest algebra alg$\mathcal{M}$ has nine distinct first and second order Bourgain algebras because all nine corresponding subnests of $\mathcal{N}_1$ are distinct. Also, because the subnests $(((\mathcal{N}_2)_r)_r)_l$, $(((\mathcal{N}_2)_r)_l)_r$, $(((\mathcal{N}_2)_\infty)_r)_l$, $(((\mathcal{N}_3)_l)_r)_l$, $(((\mathcal{N}_3)_l)_l)_r$, and $(((\mathcal{N}_3)_\infty)_l)_r$ are all distinct, it follows that alg$\mathcal{M}$ has six different third order Bourgain algebras. Moreover, one can show that each of the third order subnests $((\mathcal{M}_\star)_r)_l$ and $((\mathcal{M}_\star)_r)_l$, where $\star$ is any of r, l, or ∞, is different from all second order subnests of $\mathcal{M}$. Thus, alg$\mathcal{M}$ has the maximum complement of fifteen distinct Bourgain algebras of all orders combined as claimed.

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