

Sequences of Chords and of Parabolic Segments Enclosing Proportional Areas

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The graph of $y = x^3$ and a line tangent to it enclose an area. In [1] it is shown that the areas determined by successive such tangent lines are proportional to one another. In the left-hand side of the figure, the first tangent line encloses the region shaded in black, and the second tangent line encloses the gray region which has 16 times as much area. This motivated our consideration of similar problems for the family of parabolas $y = cx^2$. For a parabola, tangent lines enclose no such areas; thus, it was natural for us to consider instead areas determined by secant lines, and that's what we did. Specifically, we considered a parabola, and, beginning at the origin, we found a sequence of points $\{p_n\}$ on it such that the areas enclosed by the parabola and by the chords joining successive pairs of points in the sequence were proportional. The right-hand side of the figure shows the first two chords when the proportionality constant is 1. In this case, the point p_2 is chosen so that the area of the gray region is equal to the area of the black region. We then examined the lengths of these chords and the lengths of the parabolic segments joining consecutive points, as well as the ratios of such lengths. It was the limit of some of these ratios as $n \rightarrow \infty$ that we found to be surprising.

[Insert figures here side by side, as indicated on figures themselves.]

Our results are about area, arc length, and limits, concepts that are treated in a standard first-year calculus course. Occasionally we deal with expressions that seem complicated, but in fact the techniques we use are nevertheless generally simple and within the reach of first-year calculus students. We enjoyed discovering unexpected results in a simple context, and hope that students' experience with this material will be similar.

Start by choosing positive constants c and m . The parabolic curve $y = cx^2$ and the line $y = mx$ intersect at two points, $p_0 = (0, 0)$ and $p_1 = (m/c, m^2/c)$. Let A denote the area of the region enclosed by the parabolic arc between p_0 and p_1 and the chord connecting p_0 and p_1 . Then,

$$A = \int_0^{m/c} (mx - cx^2) dx = m^3/(6c^2). \quad (1)$$

Now choose a proportionality constant $k > 0$ and find a point $p_2 = (a_2, ca_2^2)$ on the parabola such that $a_2 > m/c$ and such that the region enclosed by the parabolic arc between p_1 and p_2 and the chord connecting p_1 and p_2 has an area equal to kA . Continue this process so that, once p_0, \dots, p_n have been found, then $p_{n+1} = (a_{n+1}, ca_{n+1}^2)$ is chosen in such a way that $a_{n+1} > a_n$ and that the region enclosed by the parabolic arc and by the chord connecting p_n and p_{n+1} has an area equal to $k^n A$. In other words, a_{n+1} is chosen to satisfy

$$k^n m^3 / (6c^2) = \int_{a_n}^{a_{n+1}} \{c(a_{n+1} + a_n)(x - a_n) + ca_n^2 - cx^2\} dx.$$

It follows from this that

$$a_{n+1} = (m/c)k^{n/3} + a_n$$

and, by recursion, that

$$a_n = \frac{m}{c} \sum_{j=0}^{n-1} k^{j/3} = \begin{cases} \left(\frac{m}{c}\right)n, & \text{if } k = 1, \\ \left(\frac{m}{c}\right) \left(\frac{1-k^{n/3}}{1-k^{1/3}}\right), & \text{if } k \neq 1. \end{cases} \quad (2)$$

For $0 < k < 1$, observe that $\lim_{n \rightarrow \infty} a_n = (m/c)(1 - k^{1/3})^{-1}$. For $k \geq 1$, we see that $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Denote by d_n the length of the chord and by L_n the length of the parabolic arc connecting the points p_n and p_{n+1} . It is evident that

$$d_n^2 = (a_{n+1} - a_n)^2 + (ca_{n+1}^2 - ca_n^2)^2 = (a_{n+1} - a_n)^2 [1 + c^2(a_{n+1} + a_n)^2] \quad (3)$$

and that

$$L_n = \int_{a_n}^{a_{n+1}} \sqrt{1 + 4c^2x^2} dx \quad (4)$$

for all $n \geq 0$. Two useful inequalities are

$$L_n \leq (a_{n+1} - a_n) \sqrt{1 + 4c^2a_{n+1}^2}, \quad (5)$$

which follows from (4), and

$$L_n^2 \leq [(a_{n+1} - a_n) + (ca_{n+1}^2 - ca_n^2)]^2 = (a_{n+1} - a_n)^2 [1 + c(a_{n+1} + a_n)]^2, \quad (6)$$

which follows from (4) and the fact that $\sqrt{1+t^2} < 1+t$ for all $t > 0$.

We can see immediately now from (2), (3), (5), and the obvious fact that $d_n \leq L_n$ for all n , that

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} L_n = \begin{cases} 0, & \text{if } 0 < k < 1, \\ \infty, & \text{if } k \geq 1. \end{cases}$$

This is the result one would expect from looking at a graph.

We now look at the limits of the ratios d_{n+1}/d_n , L_n/d_n , and L_{n+1}/L_n . For the first of these, (2) and (3) together yield

$$\frac{d_{n+1}}{d_n} = \frac{k^{1/3} \sqrt{1 + m^2 [Q_{n+1}(k)]^2}}{\sqrt{1 + m^2 [Q_n(k)]^2}},$$

where $Q_n(k) = k^{n/3} + 2 \sum_{j=0}^{n-1} k^{j/3}$. Using the formula for the partial sum of a geometric series and handling the cases $0 < k < 1$, $k = 1$, and $k > 1$ separately, we found that

$$\lim_{n \rightarrow \infty} \frac{d_{n+1}}{d_n} = \begin{cases} k^{1/3}, & \text{if } 0 < k < 1, \\ 1, & \text{if } k = 1, \\ k^{2/3}, & \text{if } k > 1. \end{cases}$$

Looking at a graph, we suspected that the arc and the chord would tend to be of equal lengths in the long run. Indeed, supposing $0 < k < 1$, it follows from (3), (5), and the fact that $d_n \leq L_n$, that

$$1 \leq \frac{L_n^2}{d_n^2} \leq \frac{1 + 4c^2 a_{n+1}^2}{1 + c^2(a_{n+1} + a_n)^2} \rightarrow 1$$

as $n \rightarrow \infty$, since $a_n \rightarrow (m/c)(1 - k^{1/3})^{-1}$ in this case. When $k \geq 1$, then (3), (6), and the fact that $a_n \rightarrow \infty$ give

$$1 \leq \frac{L_n^2}{d_n^2} \leq \frac{[1 + c(a_{n+1} + a_n)]^2}{1 + c^2(a_{n+1} + a_n)^2} \rightarrow \frac{c^2}{c^2} = 1$$

as $n \rightarrow \infty$.

Finally, since $L_{n+1}/L_n = (L_{n+1}/d_{n+1})(d_{n+1}/d_n)(d_n/L_n)$, we see that

$$\lim_{n \rightarrow \infty} \frac{L_{n+1}}{L_n} = \begin{cases} k^{1/3}, & \text{if } 0 < k < 1, \\ 1, & \text{if } k = 1, \\ k^{2/3}, & \text{if } k > 1. \end{cases}$$

The dependence of these limits on the value of k intrigued us. From the picture alone we hadn't been able to formulate a good conjecture for the limits of d_{n+1}/d_n and L_{n+1}/L_n , so we found the end result to be delightful and unexpected.

In the same context, one can establish similar results. For instance, when $k = 1$, for each of the chords joining successive points in the sequence $\{p_n\}_{n=0}^{\infty}$, one can consider the distance between the chord and the line parallel to the chord but tangent to the parabola. It turns out that, as $n \rightarrow \infty$, the successive distances tend to 0, and that the ratio of one such distance to the next tends to 1.

A problem that remains is the generalization of our results to integral powers of x higher than 2. It was not clear from our preliminary investigations of this question what to expect. Moreover, using a computer algebra system did not seem to clarify the situation, even for the function x^3 . Considering a more general increasing, convex curve in place of the parabola is even more problematic as such a curve is not even guaranteed to intersect a given line $y = mx$ more than once. For instance, if $m \geq 1$, the curve $y = x - 1 + e^{-x}$ intersects $y = mx$ only at the origin.

Finally, we gratefully acknowledge the journal Editor, whose many helpful suggestions improved the paper and enlightened us as well.

Reference

- [1] Herbert I. Brown, James W. Burgmeier, and David S. Dummit, Functions whose successive tangent lines enclose proportional areas, *American Mathematical Monthly* 103 (1996) 779–787.