Intersections of Tangent Lines of Exponential Functions

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Among the intriguing geometric properties of a parabola is this: if AB is any chord of the parabola and L is the tangent to the parabola parallel to AB, then the midpoint of the chord, the point of tangency of L, and the point where the tangents to the parabola at A and at B intersect are collinear. Moreover, the line they form is parallel to the axis of symmetry of the parabola. So, if we make the parabola's axis of symmetry coincide with the y-axis in a Cartesian coordinate system, then it follows that the x-coordinate of these points is the average of the x-coordinates of A and B. (See Stein [**2**] for more on this.)

In [**3**], Stenlund shows that, among all analytic curves, this property is unique to parabolas. In discussing this result, Stenlund looks at a second curve associated with a parabola. Specifically, fix a positive horizontal displacement h, and, for any chord AB where the x-coordinates of A and B differ by h, look at the point where the tangents to the parabola at A and at B intersect. The locus of all such intersections of pairs of tangents forms another parabola, as Stenlund demonstrates. Moreover, this derived parabola is a translation along the axis of symmetry of the original parabola by an amount that depends on h. Figure 1 illustrates this construction for the parabola $y = x^2$ with $h = 2$.

We wondered whether a parabola was the only curve for which this associated *derived curve* was essentially a copy of itself, and thus we were led to consider exponential functions.

Derived curves

We extend Stenlund's construction to exponential functions. For $k \neq 0$, let $g(x) := e^{kx}$. Fix $h > 0$ and let t be an arbitrary real number. The lines tangent to the graph of g at

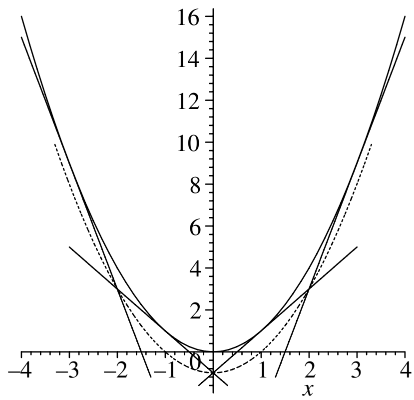

Figure 1. The parabola $y = x^2$ and its associated derived parabola $y = x^2 - 1$ corresponding to the horizontal displacement $h = 2$.

the points $(t - h/2, g(t - h/2))$ and $(t + h/2, g(t + h/2))$ are

$$T_1(x) = e^{k(t-h/2)}\left\{k(x - t + h/2) + 1\right\}$$

and

$$T_2(x) = e^{k(t+h/2)}\left\{k(x - t - h/2) + 1\right\},$$

respectively. The x-coordinate of their intersection point is

$$x_t = t - \frac{1}{k} + \left(\frac{h}{2}\right)\left(\frac{e^{kh} + 1}{e^{kh} - 1}\right),$$

so its y-coordinate is

$$y_t = \left(\frac{kh}{e^{kh} - 1}\right) e^{k(t+h/2)}.$$

(The left-hand diagram in Figure 2 illustrates this construction.) We thus have parametric equations for the associated derived curve, and we can easily eliminate t to get y_t directly as a function of x_t. After simplifying, we have

$$y_t = \left(\frac{kh}{e^{kh} - 1}\right) e^{\left(1 - \frac{kh}{e^{kh}-1}\right)} e^{kx_t}.$$

In summary, we have shown the following:

Theorem. *Let h and k be real numbers with $h > 0$ and $k \neq 0$, and let*

$$\alpha(k, h) := \left(\frac{kh}{e^{kh} - 1}\right) e^{\left(1 - \frac{kh}{e^{kh}-1}\right)}.$$

Then the derived curve for the exponential function $g(x) := e^{kx}$ is $\widetilde{g}(x) = \alpha(k, h)g(x)$.

Thus, the derived curve $\widetilde{g}$ of the exponential function g is also an exponential function, and $\widetilde{g}$ differs from g by the scaling factor $\alpha(k, h)$. Figure 2 gives an illustration.

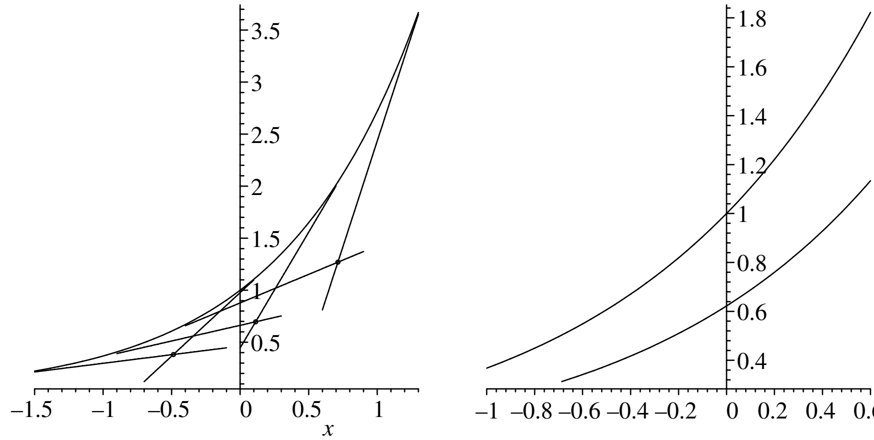

Figure 2. $g(x) := e^x$ and its associated derived curve $\widetilde{g}(x) = \alpha(1, 2)g(x)$; $\alpha(1, 2) \approx 0.6222$.

The scaling factor $\alpha(k, h)$

We now consider the behavior of the scaling function $\alpha(k, h)$. To begin, let

$$f(x) := \frac{x}{e^x - 1}, \quad \text{for } x \neq 0.$$

Observe that $f(x) > 0$ for all nonzero x, and that $\lim_{x\to 0} f(x) = 1$. Thus, we define $f(0) := 1$ to get a function that is continuous on the real line.

That $e^x - 1 > x$ for all nonzero x is a straightforward application of the Mean Value Theorem. Therefore, $0 < f(x) \leq 1$ for $x \geq 0$ and $f(x) \geq 1$ for $x \leq 0$. Moreover, an application of l'Hôpital's Rule shows that $f'(0) = -1/2$. Thus, f is differentiable on $\mathbb{R}$.

Now consider the function $p(t) := te^{(1-t)}$. Since $p'(t) = (1-t)e^{(1-t)}$, we see that p attains its absolute maximum value at $p(1) = 1$. The substitution of $f(x)$ for t shows that the function $F(x) := f(x)e^{(1-f(x))}$ is differentiable and satisfies $0 < F(x) \leq 1$ on $\mathbb{R}$. Also, $F(x) = 1$ only for $x = 0$.

Finally, let h, k, and $\alpha(k, h)$ be as in the theorem. Thus, $\alpha(k, h) = F(kh)$, whence $\lim_{h\to 0+} \alpha(k, h) = 1$ for each fixed value of k, as one would have hoped. In fact, the equation $\alpha(k, h) = F(kh)$ extends α to a differentiable function in the kh-plane that satisfies $0 < \alpha(k, h) \leq 1$ and $\alpha(k, h) = 1$ only when $kh = 0$. The level curves of the graph of α are the hyperbolas $kh =$ constant. The graph of α is shown in Figure 3.

In conclusion, we have shown that the derived curve associated with an exponential function is again an exponential function, so this sort of invariance is not unique to parabolas. Unlike the situation for parabolas, however, the derived curve of an expo-

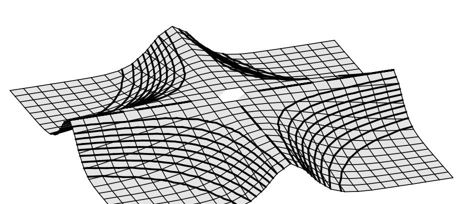

Figure 3. The graph of $\alpha(k, h)$ for $-4 \leq k, h \leq 4$. Maple does not plot $\alpha(0, 0)$, which is defined only as a limit. Note the hyperbolic level curves.

nential function is not a translation, but a rescaling of the original curve by a factor whose properties we have analyzed.

In [**4**], Wilkins uses intrinsic geometric characterizations of the conic sections to generalize the main results of [**3**]. He shows that the property of parabolas cited in the opening paragraph of this article can be expressed as an angle-bisection property that is unique to the conic sections. Lacking an intrinsic geometric formulation of the derived curve of an exponential, we have been unable to determine whether or not the exponentials are unique in possessing the property that the derived curve is a multiple of the original curve.

References

1. T. L. Heath, *The Works of Archimedes*, Dover, 1953.
2. S. Stein, *Archimedes: What Did He Do Besides Cry Eureka?*, MAA, 1999.
3. M. Stenlund, On the tangent lines of a parabola, *College Math. J.* **32** (2001) 194–196.
4. D. Wilkins, The tangent lines of a conic section, *College Math. J.* **34** (2003) 296–303.

Federal Money

The denominations are:

10 mills (*m*) make	1 cent,	*c*.
10 cents - - -	1 dime,	*d*.
10 dimes - -	1 Dollar,	*D*.
10 dollars -	1 Eagle,	*E*.

• • •

Federal money, or money of the United States, may be added, subtracted, multiplied and divided as integers or whole numbers, only separating the different denominations with a point, as fifty-nine eagles, five dollars, nine dimes, five cents, in figures 59,5,9,5: but as dollars and cents are the only denominations commonly used in accounts, the points after the eagles and dimes are omitted, as 595,95, five hundred and ninety-five dollars and ninety-five cents.

Dollars are reduced to cents, by multiplying the number of dollars by 100, or which is the same thing, by adding two ciphers to the right hand of the number of dollars, as,

In 1 dollar, how many cents?	*Answer*	100
In 6 dollars, how many cents?	*Answer*	600
In 10 dollars, how many cents?	*Answer*	1000

Cents are brought into dollars by dividing by 100, or separating the two last figures to the right hand by a point, which will be cents, and those to the left will be dollars, as,

In 225 cents, how many dollars and cents?	*Answer*	2,25
In 506 cents, how many dollars and cents?	*Answer*	5,06
In 1250 cents, how many dollars and cents?	*Answer*	12,50

Note. In writing down any number of cents less than 10, a cipher must be prefixed in the place of dimes.

The American Tutor's Assistant Revised; or,
A Compendious System of Practical Arithmetic
Printed by Joseph Crukshank, Philadelphia, 1809,
page 14.

