RESEARCH ARTICLE

On the area of a parabolic sector

Timothy G. Feeman
Dept. of Mathematical Sciences, Villanova University, Villanova, PA 19085-1699
e-mail: timothy.feeman@villanova.edu; telephone: 610-519-4693; fax: 610-519-6928
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We present a “base × height” approach to familiar results of Archimedes on the area of a parabolic sector.

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I have developed the habit, while in my department’s copy room waiting for the machine to churn out handouts, quizzes, and assignment sheets, of browsing through the excellent collection *A Century of Calculus* ([1]) that is kept on the shelves there. So it happened that I came across the article *Area of a parabolic region*, by Rozen and Sofo ([3]), wherein the authors show that the area of a region bounded by a parabola and a chord that is perpendicular to the parabola’s axis of symmetry is equal to 2/3 of the product of the length of the chord and the height of the region. This is, of course, a special case of a result of Archimedes that the area of any parabolic sector is given by 4/3 of the area of the inscribed triangle defined by the endpoints of the chord and the point on the parabola where the tangent line is parallel to the chord, or, equivalently, by 2/3 of the area of the circumscribed parallelogram. Nonetheless, the “base times height” flavor of Rozen and Sofo’s result appealed to me, and I wondered how to express the general result in those familiar terms. (See [2] for the authoritative work on Archimedes and [4] for a modern treatment of some of his results on the parabola.)

In pursuit of Archimedes

Let \( P \) be an arbitrary parabola. Let \( A \) and \( B \) denote two arbitrarily chosen points on \( P \), let \( M \) be the midpoint of the segment \( \overline{AB} \), and let \( V \) denote the “vertex” of the parabolic sector defined by \( P \) and \( \overline{AB} \). That is, \( V \) is the point on \( P \) where the tangent to the parabola is parallel to the segment \( \overline{AB} \). It is a basic fact about parabolas that the segment \( \overline{MV} \) is parallel to the axis of symmetry of the parabola. See Figure 1(a).

To aid us in measuring the area of the sector, we possess one fundamental tool that Archimedes did not: the function concept of analytic geometry. Indeed, if we denote by \( p \) the distance between the vertex of the parabola and its focus, then, by appropriate choice of coordinate axes, we may assume, without loss of generality, that \( P \) is given by the equation \( y = ax^2 \), where \( a = 1/(4p) \). Now to coordinatize everything, let the point \( V \) have coordinates \((\epsilon, a\epsilon^2)\), and let \( A \) and \( B \) have \( x \)-coordinates \( \epsilon - \delta \) and \( \epsilon + \delta \), respectively. The tangent to the parabola at \( V \) has
slope 2ae, and the line through A and B, which is parallel to that tangent, has equation

\[ y = 2aex - ae^2 + a\delta^2. \tag{1} \]

The area of the parabolic sector is determined by the integral

\[ \text{Sector Area} = \int_{\epsilon-\delta}^{\epsilon+\delta} \left\{ (2aex - ae^2 + a\delta^2) - (ax^2) \right\} \, dx = \frac{4}{3}a\delta^3. \tag{2} \]

The substitution \( a = 1/(4p) \) yields simply

\[ \text{Sector Area} = \frac{\delta^3}{3p}. \tag{3} \]

As the quantity \( \delta \) is just half of the “horizontal” displacement between the two points A and B (where “horizontal” means “perpendicular to the axis of the parabola”), this formula expresses the area of the sector entirely in terms of the intrinsic geometry of the parabola and the given chord. It is also pretty easy to memorize (if one really wished to do so).

To put a “base times height” spin on this result, observe that the y-coordinate of the point M is just the average of the y-coordinates of the points A and B. Thus,

\[ |MV| = \left( \frac{1}{2} \right) (a(\epsilon - \delta)^2 + a(\epsilon + \delta)^2) - ae^2 = a\delta^2. \tag{4} \]

If we think of 2\( \delta \) as the “horizontal” dimension of the sector and \( MV \) as the “vertical” dimension, then the area of the sector is \((2/3) \times (“horizontal”) \times (“vertical”)\).

For a different point of view, consider that the chord \( AB \) and the tangent to \( P \) at V define a parallelogram, shown in Figure 1(b), where the edges \( AD \) and \( BC \) are parallel to \( MV \).

The area of the parallelogram \( ABCD \) is given by \((2\delta) |MV| \). As in (4), \( |MV| = a\delta^2 \), so that the parallelogram has area \( 2a\delta^3 \). Comparing this to (2), we now see that the area of the parabolic sector is equal to \((2/3)\) of the area of the circumscribed parallelogram. It is easy to see that the parallelogram has twice the area of the triangle \( \triangle ABV \). So the area of the parabolic sector is equal to \((4/3)\) of the area of the inscribed triangle, as established by Archimedes millenia ago.
More “base × height”

From Figure 1(b), we see that the area of the parallelogram $ABCD$ is also given by the product $|AB||QV|$, where $Q$ denotes the point on $AB$ such that the segment $QV$ is perpendicular to $AB$. Thus, the area of the sector is $(2/3)|AB||QV|$, where $V$ denotes the point on $AB$ such that the segment $QV$ is perpendicular to $AB$. Thus, the area of the sector is

$$\text{Sector Area} = \left(\frac{2}{3}\right)|AB||QV| = \left(\frac{2}{3}\right)|AB|\cos(\theta)|MV| = \frac{4}{3}a\delta^3. \quad (5)$$

On the other hand, the line through $Q$ and $V$, being perpendicular to the tangent at $V$, has equation

$$y = -\frac{1}{2ae}(x - \epsilon) + ae^2. \quad (6)$$

The point $Q$, which is the point of intersection of these two lines, thus has the $x$-coordinate

$$x_* := \frac{4a^2\epsilon^3 - 2a^2\epsilon\delta^2 + \epsilon}{1 + 4a^2\epsilon^2}. \quad (7)$$

The $y$-coordinate of $Q$ is then $y_* := ae^2 - (x_* - \epsilon)/(2ae)$. The expression $x_* - \epsilon$ simplifies to $-(2a^2\epsilon\delta^2)/(1 + 4a^2\epsilon^2)$. Therefore, we get

$$|QV|^2 = (x_* - \epsilon)^2 + (y_* - ae^2)^2 \quad = (x_* - \epsilon)^2 + \frac{(x_* - \epsilon)^2}{4a^2\epsilon^2} \quad = \left(\frac{-2a^2\epsilon\delta^2}{1 + 4a^2\epsilon^2}\right)^2 \left(\frac{1 + 4a^2\epsilon^2}{4a^2\epsilon^2}\right) \quad = \frac{a^2\delta^4}{1 + 4a^2\epsilon^2}.$$

Recalling that $|MV| = a\delta^2$, we see that $|QV| = |MV|/\sqrt{1 + 4a^2\epsilon^2}$. The substitution $a = 1/(4p)$ yields that $1/\sqrt{1 + 4a^2\epsilon^2} = 2p/\sqrt{4p^2 + \epsilon^2}$. (This expression is, in fact, $\cos(\theta)$ by equation (5) above.)

It follows, finally, that the area of the parabolic sector determined by the chord $AB$ is given by

$$\text{Sector Area} = \frac{2}{3}|AB||MV|\frac{2p}{\sqrt{4p^2 + \epsilon^2}}. \quad (8)$$

The quantity $p$, we recall, denotes the distance between the vertex of the parabola and its focus, while $\epsilon$ represents the distance between the point $V$ and the axis of symmetry of the parabola. Thus, the formula above is expressed entirely in terms of the intrinsic geometry of the parabola itself and the chord that determines the sector.

If we think of the length $|AB|$ as the base of the parabolic sector and the length $|MV|$ as its height (though not its altitude, obviously), then we can interpret (8) as a “base times height” formula, where the expression $2p/\sqrt{4p^2 + \epsilon^2}$ serves as a correction factor for the slant of the sector. (Again, this expression is equal to
cos(θ), where θ is the angle that \overrightarrow{AB} makes with the horizontal.) If \epsilon = 0, then the sector is of the simplest type considered by Rozen and Sofo and its area really is \((2/3) \times \text{(base)} \times \text{(height)}\).

An affine insight

The integrand in (2) above is a quadratic expression and, so, defines a parabola. In fact, this parabola is "affinely equivalent" to the parabola \(P\). Specifically, define an affine transformation of the plane by

\[
T(x, y) := (x - \epsilon, y - 2a\epsilon x + a\epsilon^2),
\]

using all of the previous notations. Then \(T\) maps \(P\) to itself, since \(T(x, ax^2) = (x - \epsilon, a(x - \epsilon)^2)\). Moreover, \(T\) maps the sector shown in Figure 1(a) into a sector, determined by the horizontal chord \(T(A)T(B)\), whose vertex is the vertex of the parabola itself. (Note that \(T(V) = (0, 0)\) so that \(T\) maps the "vertex" of the sector to the vertex of \(P\).) The chord \(T(A)T(B)\) joins the points \((-\delta, a\delta^2)\) and \((\delta, a\delta^2)\). Therefore, the area of the new sector is equal to \((2/3) \times \text{(base)} \times \text{(height)}\), or \((2/3)(2\delta)(2a\delta^2) = (4/3)a\delta^3\).

Finally, recall that an affine transformation transforms areas of regions in the plane by a factor equal to the absolute value of the determinant of the linear part of the transformation. The linear part of the transformation \(T\) in (9) is the map \((x, y) \mapsto (x, y - 2a\epsilon x)\). This map has the matrix \[
\begin{bmatrix}
1 & 0 \\
-2a\epsilon & 1
\end{bmatrix}
\]
which has determinant \((1)(1) - (0)(-2a\epsilon) = 1\). Hence, the area of the sector determined by the chord \(AB\) is also equal to \((4/3)a\delta^3\), as we found in (2).

References