Ratios of volumes related to
the odd extension of a power function

Introduction

Encouraged by correspondence we have received over the past few years, we return here to a situation similar to that of the earlier papers [1], [2], [3], and [4]. In this setting, the initial observation was that the line tangent to the curve $y = x^3$ at the point $(a, a^3) \neq (0, 0)$ intersects the curve at exactly one other point. Thus, there is a well-defined region enclosed by that tangent line and the curve. A second region can be formed by drawing another tangent line to the cubic curve at the point where the first tangent line intersects the curve. We will refer to the tangent lines related in this way as successive tangent lines to the curve. That is, each successive tangent line is tangent to the curve at the other point where the previous tangent line crosses the curve. In [1], the authors began by showing that the areas of the regions enclosed by successive tangent lines are proportional to one another. Those authors then generalized that result to the odd extension of $x^\alpha$, defined below; this is an extension of the power function $x^\alpha$ to an odd function of $x$. In [2], we considered the lengths of the chords defined by the successive tangent lines to the odd extension of $x^\alpha$ and found that a similar proportionality result holds, but only in the limit as one chord is compared to the next for an infinite succession of tangent lines. Here, we look at the volumes obtained by revolving each of the enclosed regions about the tangent line that defines it and find several proportionality results that hold, as before, in the limit. We conclude with a suggestion for further work in a similar vein.

1 The odd extension of $x^\alpha$

As in [1], define the odd extension of $x^\alpha$, for each positive real number $\alpha \neq 1$, by

$$F_\alpha(x) = \begin{cases} x^\alpha, & \text{for } x \geq 0, \\ -(\neg x)^\alpha, & \text{for } x < 0. \end{cases} \quad (1)$$

The case where $\alpha = 1$ requires special treatment.

In Proposition 1 of [1], the authors prove that, when $\alpha \neq 1$ is a positive real number, then, for each $a \neq 0$, the tangent line to the curve $y = F_\alpha(x)$ at $x = a$ intersects the curve in precisely one additional point, namely the point with $x = -\omega(\alpha)a$, where $\omega(\alpha)$ is the unique positive solution to $x^\alpha - \alpha x - \alpha + 1 = 0$. Moreover, they show that $\omega(\alpha) > 1$, a fact we use below. Let $A(a)$ denote the area of the region enclosed by the graph of $F_\alpha(x)$ and its tangent line at $x = a$ between $x = a$ and $x = -\omega(\alpha)a$. In turn, the tangent line at $x = -\omega(\alpha)a$ defines a region with area $A(-\omega(\alpha)a)$. For positive $\alpha \neq 1$ and $a \neq 0$, Theorem 3 of [1] shows that the ratio, $A(-\omega(\alpha)a)/A(a)$, of the areas of these successive regions is equal to $\omega(\alpha)^{\alpha+1}$, independent of $a$. 
2 Volumes defined by successive regions

Again looking at the region enclosed by the graph of $F_\alpha(x)$ and its tangent line at $x = a$ between $x = a$ and $x = -\omega(\alpha)a$, let $V(a)$ denote the volume of the solid obtained by revolving this region about the tangent line that defines it. This solid resembles a sort of pod attached to the curve.

![Figure 1: The region bounded by the graph of $F_\alpha$ and a tangent line is revolved about the tangent line. Here $\alpha > 1$ and $a > 0$.](image)

We consider first the case where $\alpha > 1$ and abbreviate $\omega(\alpha)$ as simply $\omega$. Assuming, for the moment, that $a > 0$, then the line tangent to the graph of $F_\alpha$ at the point $(a, a\alpha)$ has equation

$$\frac{\alpha a - 1}{1 - \alpha^2a^{2\alpha-2}}x - y + (1 - \alpha)a^\alpha = 0.$$  

Thus, for each value of $t$, the square of the distance from the point $(t, F_\alpha(t))$ to this tangent line is

$$[r(t)]^2 = \frac{(\alpha a^{\alpha-1}t - F_\alpha(t) + (1 - \alpha)a^\alpha)^2}{1 + \alpha^2a^{2\alpha-2}}.$$  

(2)

This is the square of the radius of a typical “slice” in the solid, made at right angles to the axis of rotation. Figure 1 illustrates this case with $a > 0$.

To compute the width of a typical slice, first we calculate that the line through $(t, F_\alpha(t))$ and perpendicular to the tangent line intersects the tangent line at the point $(x_t, y_t)$, where

$$x_t = \frac{t + \alpha a^{\alpha-1}F_\alpha(t) + \alpha(\alpha - 1)a^{2\alpha-1}}{1 + \alpha^2a^{2\alpha-2}}$$ and $$y_t = \alpha a^{\alpha-1}x_t + (1 - \alpha)a^\alpha.$$  

(3)

Thus, as $t$ varies from $-\omega a$ to $a$, the appropriate segment of the tangent line is parameterized by $(x_t, y_t)$. For this parameterization,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx_t}{dt}\right)^2 + \left(\frac{dy_t}{dt}\right)^2} = \frac{1 + \alpha a^{\alpha-1}F_\alpha'(t)}{\sqrt{1 + \alpha^2a^{2\alpha-2}}} = \begin{cases} \frac{1 + \alpha^2a^{\alpha-1}(-t)^{\alpha-1}}{\sqrt{1 + \alpha^2a^{2\alpha-2}}}, & \text{if } t < 0, \\ \frac{1 + \alpha^2a^{\alpha-1}t^{\alpha-1}}{\sqrt{1 + \alpha^2a^{2\alpha-2}}}, & \text{if } t \geq 0. \end{cases}$$  

(4)

The width of a slice is then $ds$, and the volume of the solid is $V(a) = \int_{t=-\omega a}^{a} \pi [r(t)]^2 ds$, or

$$V(a) = \left\{ \frac{\pi}{(1 + \alpha^2a^{2\alpha-2})^{3/2}} \right\} \int_{t=-\omega a}^{a} \left(\alpha a^{\alpha-1}t - F_\alpha(t) + (1 - \alpha)a^\alpha\right)^2 \left(1 + \alpha a^{\alpha-1}F_\alpha'(t)\right) dt.$$  

(5)

In keeping with the definition of $F_\alpha$, this integral can be broken up into two integrals, one for $-\omega a \leq t \leq 0$ and one for $0 \leq t \leq a$. The integrands are not particularly difficult, involving only
power functions of \( t \). But the limits of integration involve \( \omega \), so the property \( \omega^\alpha - \alpha \omega - \alpha + 1 = 0 \) is needed to simplify the results of the integration.

Also, it follows from the symmetry of the graph of \( F_\alpha \) that \( V(-a) = V(a) \). That is, the value of \( V(a) \) depends only on \( |a| \) (and of course on \( \alpha \)). Indeed, after simplification, we find

\[
V(a) = \left( \frac{\pi}{3} \right) \left( \frac{\alpha^2(\omega + 1)^2(\alpha - 1)^2(2\alpha\omega + 2\alpha + 2\omega + 5)}{7\alpha^2 + 2\alpha^3 + 7\alpha + 2} \right) \frac{|a|^{2\alpha+1}}{\sqrt{1 + \alpha^2a^{2\alpha-2}}}.
\]

(6)

For example, if \( \alpha = 3 \), then \( \omega = 2 \) and \( V(a) = \frac{729\pi|a|^7}{35\sqrt{1+9a^2}} \).

If we consider now two successive solids, one defined by the tangent line to the graph of \( F_\alpha \) at \( x = a \) and the next by the tangent line at \( x = -\omega a \), then we find the ratio of the corresponding volumes to be

\[
\frac{V(-\omega a)}{V(a)} = \frac{|-\omega a|^{2\alpha+1}}{\sqrt{1 + \alpha^2(-\omega a)^{2\alpha-2}}} \frac{\sqrt{1 + \alpha^2a^{2\alpha-2}}}{|a|^{2\alpha+1}} = \omega^{2\alpha+1} \left( \frac{1 + \alpha^2a^{2\alpha-2}}{1 + \alpha^2(-\omega a)^{2\alpha-2}} \right)^{1/2}.
\]

(7)

To create a sequence of successive solids, we fix a value \( a_0 \neq 0 \) and, for each natural number \( k \), define \( a_k = (-\omega)^k a_0 \). The line tangent to the graph of \( F_\alpha \) at \( x = a_k \) intersects the graph of \( F_\alpha \) again at the point where \( x = a_{k+1} \), thus defining a region enclosed between the curve and the tangent line. When revolved about the tangent line, this region generates a solid whose volume, \( V(a_k) \), is given by the formula (6). Comparing one such volume to the next, define \( R_k = V(a_{k+1})/V(a_k) \) for each \( k \geq 0 \). Using formula (7), it follows that

\[
R_k = \omega^{2\alpha+1} \left( \frac{1 + \alpha^2(\omega^k)^{2\alpha-2}a_0^{2\alpha-2}}{1 + \alpha^2(\omega^{k+1})^{2\alpha-2}a_0^{2\alpha-2}} \right)^{1/2}.
\]

(8)

Since \( \alpha > 1 \) (so that \( 2\alpha - 2 > 0 \)) and \( \omega > 1 \), it follows that

\[
\lim_{k \to \infty} R_k = \omega^{2\alpha+1} \left( \frac{1}{\omega^{\alpha-1}} \right) = \omega^{\alpha+2}, \text{ for } \alpha > 1.
\]

(9)

Notice that the limit does not depend upon the starting point \( a_0 \).

For the case where \( 0 < \alpha < 1 \), notice that the graph of \( y = F_\alpha(x) \) is the reflection about the line \( y = x \) of the graph of \( y = F_{1/\alpha}(x) \). Therefore, the limit of the ratio \( R_k(\alpha) \) coincides with the limit of \( R_k(1/\alpha) \), or \( \omega(1/\alpha)^{2+1/\alpha} \) according to equation (9). By Theorem 4(b) of [1], we have \( \omega(1/\alpha)^{1/\alpha} = \omega(\alpha) \). It follows, then, that

\[
\omega(1/\alpha)^{2+1/\alpha} = \omega(\alpha)\omega(1/\alpha)^2 = \omega(\alpha)\omega(\alpha)^{2\alpha} = \omega(\alpha)^{2\alpha+1}.
\]

Hence,

\[
\lim_{k \to \infty} R_k(\alpha) = \omega(\alpha)^{2\alpha+1}, \text{ for } 0 < \alpha < 1.
\]

(10)

Consider now the case where \( \alpha = 1 \). Theorem 4(a) of [1] shows that \( \lim_{\alpha \to 1} \omega(\alpha) = \beta \), where \( \beta \) is the unique real root of the equation \( \beta \ln \beta = \beta + 1 \). Also, the proof of Theorem 6 of [1] demonstrates that the function

\[
F_1(x) = \begin{cases} 
  x \ln(|x|), & \text{for } x \neq 0, \\
  0, & \text{for } x = 0,
\end{cases}
\]

(11)

which is the extension of \( x \ln(x) \) to an odd function defined on the whole real line, has properties that are analogous to those of the other \( F_\alpha \). Indeed, the line tangent to the graph of \( y = F_1(x) \) at
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$x = a$, for some $a \neq 0$, intersects the graph again at the point $(-\beta a, -\beta a \ln |\beta a|)$ and, moreover, the areas of the regions enclosed by successive tangent lines are proportional, with ratio equal to $\beta^2$. Defining $\omega(1) = \beta$, we see that this agrees with the result for $F_\alpha$ with $\alpha \neq 1$. Figure 2 illustrates the case $\alpha = 1$ with $a > 0$.

Figure 2: The case $\alpha = 1$ presents an interesting twist. In a portion of the solid, several slices overlap; but using a signed version of $ds$ ensures that we get the correct volume. Here $a > 0$.

An analysis similar to that for the case $\alpha > 1$ shows that, with $a_0 \neq 0$ and $a_k = (-\beta)^k a_0$ for each natural number $k$, the volume $V(a_k)$ of the corresponding solid of revolution is given by

$$V(a_k) = \left[ \int_{a_{k+1}}^{a_k} \pi \left( (\ln(|a_k|) + 1)t - F_1(t) - a^2 \cdot (1 + \ln(|a_k|) + 1)F_1'(t) \right) \right] \frac{(4\beta^3 + 15\beta^2 + 18\beta + 7)}{(1 + (\ln(|a_k|) + 1))^2} \beta^{3k} \frac{\beta^{3k}}{\sqrt{1 + (\ln(\beta^k|a_0|) + 1)^2}},$$

where, as one would expect, the relation $\beta \ln(\beta) = \beta + 1$ has been used to simplify the results of the integration.

The ratio of successive volumes is then

$$R_k = \frac{V(a_{k+1})}{V(a_k)} = \left( \beta^3 \left( \frac{1 + (\ln(\beta^k|a_0|) + 1)^2}{1 + (\ln(\beta^{k+1}|a_0|) + 1)^2} \right) \right)^{1/2}.$$ (12)

Since $\ln(\beta^n) = n \ln(\beta) = n \left( \frac{\beta + 1}{\beta} \right)$ for any natural number $n$, we see that

$$\lim_{k \to \infty} R_k = \beta^3.$$ (13)

With $\alpha = 1$ and $\omega(1) = \beta$ this agrees with both of the formulas (9) and (10).

In sum, we have established the following result.

**Proposition 1.** For each positive real number $\alpha$, let the function $F_\alpha$ be as defined above. For $\alpha \neq 1$, let $\omega(\alpha)$ be the unique positive zero of $x^\alpha - \alpha x - \alpha + 1$, and define $\omega(1) = \beta$, the unique real root of the equation $\beta \ln(\beta) = \beta + 1$. For a fixed $\alpha > 0$, an arbitrary real number $a_0 \neq 0$, and an integer $k \geq 0$, let $a_k = (-\omega(\alpha))^k a_0$, and let $V(a_k)$ denote the volume of the solid formed when the region bounded by the graph of $F_\alpha$ and the line tangent to this graph at $x = a_k$ is revolved about the tangent line. Then

$$\lim_{k \to \infty} \frac{V(a_{k+1})}{V(a_k)} = \begin{cases} \omega(\alpha)^{\alpha+2}, & \text{for } \alpha \geq 1, \\ \omega(\alpha)^{2\alpha+1}, & \text{for } 0 < \alpha \leq 1. \end{cases}$$ (14)
3 Rates of convergence

Having determined the value of the limit of the ratio $V(a_{k+1})/V(a_k)$ in terms of the value of $\alpha$, we consider now the rate at which the ratio converges.

Beginning with the case $\alpha > 1$, let

$$Q_k = \omega^{\alpha+2} - V(a_{k+1})/V(a_k),$$

where $k \geq 0$ is an integer and $\omega = \omega(\alpha)$. According to Proposition 1, $Q_k \to 0$ as $k \to \infty$. The rate of this convergence can be measured by the relative error $Q_{k+1}/Q_k$.

Referring to (12), we get that

$$Q_k = \left(\omega^{\alpha+2}\right) \left(\frac{1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2}{1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2} - \omega^{\alpha-1} \left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2}ight),$$

whence, $Q_{k+1}/Q_k =$

$$\frac{1 + \alpha^2(\omega^{k+2})2a^2 - 2a_0^2a^2 - 1/2}{1 + \alpha^2(\omega^{k+2})2a^2 - 2a_0^2a^2 - 1/2} - \omega^{\alpha-1} \left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2}.$$  \hspace{1cm} (16)

The first factor in (17) converges to the limit $\omega^{1-\alpha}$ as $k \to \infty$. Multiplying and dividing the second factor in (17) both by

$$\left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2} + \omega^{\alpha-1} \left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2}$$

and by

$$\left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2} + \omega^{\alpha-1} \left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2}$$

yields in its place the factor

$$\frac{1 + \alpha^2(\omega^{k+2})2a^2 - 2a_0^2a^2 - 1/2}{1 + \alpha^2(\omega^{k+2})2a^2 - 2a_0^2a^2 - 1/2} + \omega^{\alpha-1} \left[1 + \alpha^2(\omega^{k+1})2a^2 - 2a_0^2a^2 - 1/2\right]^{1/2}.$$  \hspace{1cm} (17)

This factor approaches the limiting value $\omega^{1-\alpha}$ as $k \to \infty$. Thus,

$$\lim_{k \to \infty} \frac{Q_{k+1}}{Q_k} = \omega^{1-\alpha} \cdot \omega^{1-\alpha} = \omega^{2-2\alpha}, \text{ for } \alpha > 1.$$  \hspace{1cm} (18)

Since $\alpha > 1$ and $\omega > 1$, it follows that $\omega^{2-2\alpha} < 1$. By the ratio test, the series $\sum_{k\geq0} Q_k$ converges. For instance, if $\alpha = 3$, then $\omega = 2$ and the series $\sum Q_k$ converges at a rate comparable to that of the geometric series for $\omega^{2-2\alpha} = 1/16$.

When $0 < \alpha < 1$, then, as before, the symmetry of the graphs of $F_\alpha$ and $F_{1/\alpha}$ shows that

$$\lim_{k \to \infty} Q_{k+1}/Q_k = \omega(1/\alpha)^{2-2/\alpha}.$$  \hspace{1cm} (19)

Again using the fact that $\omega(1/\alpha) = \omega(\alpha)^{\alpha}$, we see that

$$\lim_{k \to \infty} Q_{k+1}/Q_k = \omega(\alpha)^{2\alpha-2}, \text{ for } 0 < \alpha < 1.$$  \hspace{1cm} (19)
With $0 < \alpha < 1$ and $\omega(\alpha) > 1$, we get that $\omega^{2\alpha-2} < 1$ and, again by the ratio test, the series $\sum_{k\geq0} Q_k$ converges. For instance, if $\alpha = 1/3$, then $\omega = 8$ and the series $\sum Q_k$ converges at a rate comparable to that of the geometric series for $\omega^{2\alpha-2} = 1/16$.

When $\alpha = 1$, then $\omega(1) = \beta$ and $2 - 2\alpha = 0$, so the previous work suggests that $\lim Q_{k+1}/Q_k$ will equal $\beta^0 = 1$, in which case the ratio test would be inconclusive for determining the convergence of the series $\sum Q_k$. Indeed, we can see from (12) that

$$Q_k = \beta^3 - R_k = \left(\beta^3\right) \frac{\left[1 + (\ln(\beta^{k+1}|a_0|) + 1)^2\right]^{1/2} - \left[1 + (\ln(\beta^k|a_0|) + 1)^2\right]^{1/2}}{\left[1 + (\ln(\beta^{k+1}|a_0|) + 1)^2\right]^{1/2}}.$$  \hspace{1cm} (20)

Multiplying and dividing by

$$\left[1 + (\ln(\beta^{k+1}|a_0|) + 1)^2\right]^{1/2} + \left[1 + (\ln(\beta^k|a_0|) + 1)^2\right]^{1/2},$$

and then simplifying using the properties of the natural logarithm and the relation $\beta \ln(\beta) = \beta + 1$, we see that the series $\sum Q_k$ diverges, in this case by virtue of a limit comparison test with the harmonic series. That is, when $\alpha = 1$, the rate of convergence of the ratio $V(a_{k+1})/V(a_k)$ to its limit $\beta^3$ is comparable to the rate at which $1/k$ converges to 0.

To summarize, we have demonstrated the following.

**Proposition 2.** For each positive real number $\alpha \neq 1$, let $\omega(\alpha)$ be the unique positive zero of $x^\alpha - \alpha x - \alpha + 1$, and define $\omega(1) = \beta$, the unique real root of the equation $\beta \ln(\beta) = \beta + 1$. For a fixed $\alpha > 0$ and an integer $k \geq 0$, let $Q_k = \lim_{n \to \infty} (V(a_{n+1})/V(a_n)) - (V(a_{k+1})/V(a_k))$; see (14). Then

$$\lim_{k \to \infty} \frac{Q_{k+1}}{Q_k} = \begin{cases} \omega(\alpha)^{2-2\alpha}, & \text{for } \alpha > 1, \\ 1, & \text{for } \alpha = 1, \\ \omega(\alpha)^{2\alpha-2}, & \text{for } 0 < \alpha < 1. \end{cases}$$  \hspace{1cm} (21)

In particular, because $\omega(\alpha) > 1$ for all $\alpha$, it follows from the ratio test that the series $\sum Q_k$ converges when $\alpha \neq 1$. When $\alpha = 1$, the series $\sum Q_k$ diverges by limit comparison with the harmonic series.

### 4 Question

We conclude with a question that we have not investigated. In [3] and [4], we started with a parabola and constructed a sequence of regions for which the areas of successive regions were in proportion. What can be said about the volumes of the solids obtained by revolving those regions about the chords that define them?

### References


TIMOTHY G. FEEMAN and OSVALDO MARRERO

*Villanova University, Villanova, PA 19085–1699*

e-mails: timothy.feeman@villanova.edu and osvaldo.marrero@villanova.edu