Lengths of successive chords of the odd extension of a power function

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Introduction

A line tangent to the curve $y = x^3$ at the point $(a,\, a^3) \neq (0,\, 0)$ intersects the curve at exactly one other point. Thus, there is a well defined area enclosed by that tangent line and the curve. In the article [BrBuD], the authors began by showing that such areas enclosed by successive tangent lines are proportional to one another. Those authors then generalized this result to the odd extension of x^α, defined below; this is an extension of the power function x^α to an odd function of x. In this note, we consider the lengths of the chords defined by the successive tangent lines to the odd extension of x^α and find that a similar proportionality result holds, but only in the limit as the number of successive steps approaches infinity.

This topic has a nice geometric flavor. Moreover, its level is that of first-year Calculus, making it accessible to a wide audience. This material could be used, for example, by teachers as projects for their students. Computational experimentation could be appropriate to this investigation, though we have not taken that approach here. Some computing could lead students to make conjectures that they would then want to prove or disprove. Also, students can see from this article that there are unanswered questions in mathematics even at an elementary level. Indeed, we conclude the paper with a suggestion for further work on a question that we were unable to resolve. All of this is beneficial for students.

The odd extension of x^α

As in [BrBuD], define the odd extension of x^α, for each positive real number $\alpha \neq 1$, by

$$F_\alpha(x) := \begin{cases} x^\alpha, & \text{for } x \geq 0\ , \\ -(-x)^\alpha, & \text{for } x < 0\ . \end{cases} \tag{1}$$

When $\alpha = 1$, the function F_1 will be defined below, as this case requires special teatment.

In Proposition 1 of [BrBuD], the authors prove that, when $\alpha \neq 1$ is a positive real number, then, for each $a \neq 0$, the tangent line to the curve $y = F_\alpha(x)$ at $x = a$ intersects the curve in precisely one additional point, namely the point with $x = -\omega(\alpha)a$, where $\omega(\alpha)$ is the unique positive solution to $x^\alpha - \alpha x - \alpha + 1 = 0$. Moreover, they show that $\omega(\alpha) > 1$, a fact we use below. Let $A(a)$ denote the area of the region enclosed by the graph of $F_\alpha(x)$ and its tangent line at $x = a$ between $x = a$ and $x = -\omega(\alpha)a$. For positive $\alpha \neq 1$ and $a \neq 0$, Theorem 3 of [BrBuD] shows that the ratio of the successive areas $A(-\omega(\alpha)a)/A(a) = \omega(\alpha)^{\alpha+1}$, independent of a.

Lengths of successive chords

Turning now to an examination of the lengths of the chords defined by the successive tangent lines of the graph of $F_\alpha(x)$ for positive $\alpha \neq 1$, begin by fixing $a \neq 0$ and setting $x_n := (-1)^n\omega(\alpha)^n a$, for each integer $n \geq 0$. Define p_n to be the point $(x_n,\, F_\alpha(x_n))$, and denote by $d_n(\alpha)$ the distance between the successive points p_n and p_{n+1}. Thus, $d_n(\alpha)$ is the length of the segment of the line tangent to the graph of $F_\alpha(x)$ at $x = x_n$ lying between the point of tangency, p_n, and the point, p_{n+1}, where this tangent line again crosses the curve. Computationally, we have

$$\begin{aligned} d_n(\alpha)^2 &= [\{\omega(\alpha)^{n+1} + \omega(\alpha)^n\}a]^2 + ([\{\omega(\alpha)^{n+1}\}^\alpha + \{\omega(\alpha)^n\}^\alpha]|a|^\alpha)^2 \\ &= [\{\omega(\alpha) + 1\}\omega(\alpha)^n a]^2 + [\{\omega(\alpha)^\alpha + 1\}\omega(\alpha)^{n\alpha}|a|^\alpha]^2. \end{aligned}$$

To evaluate the limit of the ratio $r_n(\alpha) := d_{n+1}(\alpha)/d_n(\alpha)$ of the lengths of successive chords, first consider the case where $\alpha > 1$. Dividing numerator and denominator of $r_n(\alpha)^2$ by $\omega(\alpha)^{2(n+1)\alpha}$, and using the facts that $(\alpha-1)>0$ and $\omega(\alpha)>1$, we see that

$$\begin{aligned}\lim_{n\to\infty} r_n(\alpha)^2 &= \lim_{n\to\infty}\frac{\left[\dfrac{\{\omega(\alpha)+1\}a}{\omega(\alpha)^{(n+1)(\alpha-1)}}\right]^2+[\{\omega(\alpha)^\alpha+1\}|a|^\alpha]^2}{\left[\dfrac{\{\omega(\alpha)+1\}a}{\omega(\alpha)^{\alpha+n(\alpha-1)}}\right]^2+\left[\dfrac{\{\omega(\alpha)^\alpha+1\}|a|^\alpha}{\omega(\alpha)^\alpha}\right]^2}\\ &= \frac{[\{\omega(\alpha)^\alpha+1\}|a|^\alpha]^2}{\left[\dfrac{\{\omega(\alpha)^\alpha+1\}|a|^\alpha}{\omega(\alpha)^\alpha}\right]^2}\\ &= \omega(\alpha)^{2\alpha}.\end{aligned}$$

That is,

$$\lim_{n\to\infty}\frac{d_{n+1}(\alpha)}{d_n(\alpha)}=\omega(\alpha)^\alpha \tag{2}$$

for $\alpha>1$.

For the case where $0<\alpha<1$, observe that the graph of $F_\alpha(x)$ is the reflection about the line $y=x$ of the graph of $F_{1/\alpha}(x)$ so that the limit of the ratio $r_n(\alpha)$ coincides with the limit of $r_n(1/\alpha)$, or $\omega(1/\alpha)^{1/\alpha}$ according to equation (2). From Theorem 4(b) of [BrBuD], we have $\omega(1/\alpha)^{1/\alpha}=\omega(\alpha)$ and, hence,

$$\lim_{n\to\infty} r_n(\alpha)=\omega(\alpha)$$

for $0<\alpha<1$.

To address the case where $\alpha=1$, we first note that Theorem 4(a) of [BrBuD] shows that $\lim\limits_{\alpha\to1}\omega(\alpha)=\beta$, where β is the unique real root of the equation $\beta\ln\beta=\beta+1$. Moreover, the proof of Theorem 6 of [BrBuD] reveals that the function

$$F_1(x):=\begin{cases} x\ln x, & \text{for } x>0\ ,\\ 0, & \text{for } x=0\ ,\\ x\ln(-x), & \text{for } x<0\ ,\end{cases} \tag{3}$$

which is the extension of $x\ln x$ to an odd function defined on the whole real line, has the property that the areas of the regions enclosed by successive tangent lines are proportional, with ratio equal to β^2. Defining $\omega(1):=\beta$, this is consistent with the result for $F_\alpha(x)$ with $\alpha\neq1$.

Looking at chord lengths, one can check that the line tangent to the graph of $F_1(x)$ at $x=a$, for some $a\neq0$, intersects the graph again at the point $(-\beta a,\,-\beta a\ln|\beta a|)$. For each integer $n\geq0$, define $x_n:=(-1)^n\beta^n a$, let p_n be the point $(x_n,\,F_1(x_n))$, and take $d_n(1)$ to be the length of the chord joining p_n and p_{n+1}. Thus,

$$\begin{aligned} d_n(1) &= \sqrt{(\beta^{n+1}a+\beta^n a)^2+(\beta^{n+1}a\ln|\beta^{n+1}a|+\beta^n a\ln|\beta^n a|)^2}\\ &= \beta^n|a|\sqrt{(\beta+1)^2+\{\beta(\ln\beta+\ln|\beta^n a|)+\ln|\beta^n a|\}^2}\\ &= \beta^n|a|\sqrt{(\beta+1)^2+\{\beta\ln\beta+(\beta+1)\ln|\beta^n a|\}^2}\\ &= \beta^n|a|(\beta+1)\sqrt{1+(1+\ln|\beta^n a|)^2},\end{aligned}$$

where we have used the fact that $\beta\ln\beta=\beta+1$. It follows from this that

$$\frac{d_{n+1}(1)}{d_n(1)}=\beta\sqrt{\frac{1+\{1+(n+1)\ln\beta+\ln|a|\}^2}{1+(1+n\ln\beta+\ln|a|)^2}}$$

and, hence, that

$$\begin{aligned}\lim_{n\to\infty}\frac{d_{n+1}(1)}{d_n(1)} &= \beta\cdot\lim_{n\to\infty}\sqrt{\frac{1+\{1+(n+1)\ln\beta+\ln|a|\}^2}{1+(1+n\ln\beta+\ln|a|)^2}}\\ &= \beta\cdot\lim_{n\to\infty}\sqrt{\frac{1/n^2+\{1/n+(1+1/n)\ln\beta+(\ln|a|)/n\}^2}{1/n^2+\{1/n+\ln\beta+(\ln|a|)/n\}^2}}\\ &= \beta. \end{aligned} \tag{4}$$

In sum, we have established the following result.

Proposition. *For each positive real number α, let the function $F_\alpha(x)$ be as defined above. For $\alpha \neq 1$, let $\omega(\alpha)$ be the unique positive zero of $x^\alpha - \alpha x - \alpha + 1$, and define $\omega(1) := \beta$, the unique real root of the equation $\beta\ln\beta = \beta + 1$. For a fixed $\alpha > 0$, an arbitrary real number $a \neq 0$, and an integer $n \geq 0$, let $x_n := (-1)^n\omega(\alpha)^n a$, let p_n be the point $(x_n, F_\alpha(x_n))$, and let $d_n(\alpha)$ be the distance between the points p_n and p_{n+1}. Then*

$$\lim_{n\to\infty}\frac{d_{n+1}(\alpha)}{d_n(\alpha)} = \begin{cases} \omega(\alpha)^\alpha, & \text{for } \alpha \geq 1\ , \\ \omega(\alpha), & \text{for } 0 < \alpha \leq 1\ . \end{cases} \tag{5}$$

An open problem and acknowledgements

It would be interesting to examine the lengths of the segments of the curve itself, rather than the chords, lying between the successive points p_n. It was not clear from our preliminary investigations of this question what sort of result one might expect.

Finally, we would like to thank the referees for their helpful suggestions for improving our paper.

Reference

[BrBuD] Herbert I. Brown, James W. Burgmeier, and David S. Dummit, *Functions whose successive tangent lines enclose proportional areas*, Amer. Math. Monthly, **103** (1996), 779–787.