Affine Transformations, Polynomials, and Proportionality

Timothy G. Feeman and Osvaldo Marrero

In the articles [**1**] and [**3**], standard tools and techniques of calculus are used to establish a variety of proportionality results concerning areas defined by the lines tangent to a cubic curve, and by the lengths of certain arcs of a parabola, where the arcs themselves are determined by an area-proportionality criterion. We demonstrate here that these results can be viewed as consequences of some basic facts about affine transformations in the plane.

1. AFFINE TRANSFORMATIONS. A splendid source of information about affine transformations is Appendix A of [**2**]; here are the properties we need.

Definition. An *affine transformation* of the plane is a function of the form $T(x, y) = (ax + by + e, cx + dy + f)$, where a, b, c, d, e, and f are constants and $ad - bc$ is not 0.

Properties. [**2**, Appendix A] An affine transformation T has the following properties:

(i) T takes lines to lines and parallel lines to parallel lines; that is, if $\mathcal{L}$ is a line, then $T(\mathcal{L})$ is also a line, and, if two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same slope, then the lines $T(\mathcal{L}_1)$ and $T(\mathcal{L}_2)$ also have the same slope (not necessarily the same slope as the original lines).

(ii) Segments of parallel lines are stretched or compressed in length by the same factor, which depends only on the slope of the segments; that is, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are segments of parallel lines, then there is a constant α, depending only on the slope of the segments, such that $|T(\mathcal{S}_1)| = \alpha\,|\mathcal{S}_1|$ and $|T(\mathcal{S}_2)| = \alpha\,|\mathcal{S}_2|$.

(iii) T transforms areas by the factor $|ad - bc|$; that is, if $\mathcal{R}$ is any region in the plane, then $\text{area}(T(\mathcal{R})) = |ad - bc|\,\text{area}(\mathcal{R})$.

(iv) An affine transformation of the form $T(x, y) = (ax + e, 2aex + a^2y + e^2)$ (so that $b = 0$, $c = 2ae$, $d = a^2$, and $f = e^2$) maps the parabola with equation $y = x^2$ onto itself. Moreover, T maps vertical lines to vertical lines and changes the lengths of segments of vertical lines by the factor a^2; that is, if $\mathcal{S}$ is any segment of a vertical line, then $T(\mathcal{S})$ is also vertical and $|T(\mathcal{S})| = a^2\,|\mathcal{S}|$.

2. AREA PROPORTIONALITY FOR CUBICS. Let us now look at the proportionality results of [**1**]. The line tangent to the curve $y = x^3$ at the point (a, a^3), with a nonzero, intersects the curve at one other point, with coordinates $(-2a, -8a^3)$. The line and the curve thus enclose a region $\mathcal{R}(a)$ with area $\mathcal{A}(a)$. Repeat this process, drawing the line tangent to the curve at $(-2a, -8a^3)$. This line intersects the curve again at $(4a, 64a^3)$ and encloses a region $\mathcal{R}(-2a)$ with area $\mathcal{A}(-2a)$. In [**1**] it is shown that the ratio $\mathcal{A}(-2a)/\mathcal{A}(a)$ of these areas is equal to 16, independent of the choice of a. We can prove this using affine transformations.

Lemma. $\mathcal{A}(-2a)/\mathcal{A}(a) = 16$, for all nonzero values of a.

Proof. For any constant k, the affine transformation $T(x, y) = (kx, k^3 y)$ maps the curve $y = x^3$ onto itself, since $T(x, x^3) = (kx, (kx)^3)$. By property (iii), T changes areas by a factor of k^4. Taking $k = -2$, T also has the property that $T(a, a^3) = (-2a, -8a^3)$, thus mapping any point on the cubic to the point where the tangent line there again intersects the cubic. Hence, the region $\mathcal{R}(a)$ is transformed into the region $\mathcal{R}(-2a)$. It follows that $\mathcal{A}(-2a)/\mathcal{A}(a) = k^4 = 16$ as claimed. ∎

Similarly, if $f(x)$ is any cubic polynomial with real coefficients and if $(a, f(a))$ is not the inflection point on the graph of $y = f(x)$, then the line tangent to the graph at $(a, f(a))$ will intersect the graph again at one other point. The tangent line at this second point also intersects the graph in one other place. Let $\rho(a)$ denote the ratio of the areas of the two regions defined by these successive tangent lines. The following result holds.

Theorem 1. [**1**, Theorem 1] Let $f(x)$ be any cubic polynomial with real coefficients. Then $\rho(a) = 16$ provided that $(a, f(a))$ is not the inflection point on the graph of $y = f(x)$.

Proof. Let $f(x) = Ax^3 + Bx^2 + Cx + D$ be an arbitrary cubic polynomial with real coefficients. The affine transformation

$$T(x, y) = \left(x - \frac{B}{3A}, \left(C - \frac{B^2}{3A}\right)x + Ay + \frac{2B^3}{27A^2} - \frac{BC}{3A} + D\right)$$

transforms the graph of $y = x^3$ into the graph of $y = f(x)$ and maps the inflection point $(0, 0)$ of the first graph to the inflection point of the second graph. Moreover, by the properties of affine transformations, T transforms successive tangent lines on one graph to those on the other graph, and transforms areas of all regions by a factor of $|A|$. Hence, *ratios* of areas of corresponding regions are the same for both graphs, namely 16. ∎

3. AFFINE EQUIVALANCE OF POLYNOMIALS. To examine the effect of affine transformations on polynomial curves, we begin with a definition.

Definition. Two real polynomials, p and q, of the same degree n are *affinely equivalent* if there exists an affine transformation T such that $T(x, p(x))$ has the form $(\tilde{x}, q(\tilde{x}))$ for every real number x. Such a T is called an *affine equivalence* between p and q.

For example, in the proof of Theorem 1 we saw that every cubic polynomial is affinely equivalent to x^3. Similarly, if $q(x) = Ax^2 + Bx + C$, then $q(x - B/(2A)) = Ax^2 - B^2/(4A) + C$. Therefore, the affine transformation $T(x, y) = (x - B/(2A), Ay - B^2/(4A) + C)$ transforms the parabola $y = x^2$ into the graph of q. Thus, every quadratic polynomial is affinely equivalent to x^2, and, hence, to every other quadratic.

For polynomials of degree $n \geq 4$, we begin with a simplification. Observe that, if $p(x) = \sum_{j=0}^{n} \alpha_j x^j$, then the polynomial $p_1(x) = (1/\alpha_n)\{p(x - \alpha_{n-1}/(n\alpha_n)) - p(-\alpha_{n-1}/(n\alpha_n))\}$ is a translation and dilation of p and, hence, is affinely equivalent to p. Moreover, p_1 is monic, has no x^{n-1} term, and satisfies $p_1(0) = 0$. In light of this, we investigate the affine equivalence of two polynomials, $p(x) = \sum_{j=0}^{n} \alpha_j x^j$ and $q(x) = \sum_{j=0}^{n} \beta_j x^j$, where $n \geq 4$, $\alpha_n = \beta_n = 1$, $\alpha_{n-1} = \beta_{n-1} = 0$, and $\alpha_0 = \beta_0 = 0$.

Let $T(x, y) = (ax + by + e, cx + dy + f)$ be an affine transformation with $ad - bc \neq 0$. We seek conditions on the coefficients so that $q(ax + bp(x) + e) = cx + dp(x) + f$ for all x. The left-hand side of this equation has a leading term of $(bx^n)^n$ while the leading term on the right-hand side is dx^n. It follows that $b = 0$ and that the remaining coefficients satisfy $q(ax + e) = cx + dp(x) + f$. Equating the leading coefficients on both sides of this condition yields the equation $d = a^n$, while comparing the coefficients of x^{n-1} results in the requirement that $na^{n-1}e = 0$. As $ad - bc \neq 0$, but $b = 0$, it follows that $a \neq 0$. Therefore, $e = 0$. This simplifies the condition on T to

$$q(ax) = cx + dp(x) + f. \tag{1}$$

Computing the first and second derivatives of (1) yields

$$aq'(ax) = c + dp'(x) \quad \text{and} \quad a^2q''(ax) = dp''(x). \tag{2}$$

Substituting $x = 0$ into (1) and (2), we get

$$q(0) = f, \quad aq'(0) = c + dp'(0), \quad \text{and} \quad a^2q''(0) = dp''(0). \tag{3}$$

In terms of the coefficients of p and q, we thus have $b = e = f = 0$, $d = a^n$, $c = a\beta_1 - d\alpha_1$, and $\beta_2 = a^{n-2}\alpha_2$. If this last condition can be solved for a, then it also is possible to compute c and d and, hence, the polynomials p and q are affinely equivalent.

In order to solve $\beta_2 = a^{n-2}\alpha_2$ for a, there are several cases to consider. First, if either α_2 or β_2 is 0, then the other must be 0 as well. If neither of these coefficients is 0, and if n is odd, then we can solve for $a = (\beta_2/\alpha_2)^{1/(n-2)}$. If β_2 and α_2 are nonzero and n is even, then $a^{n-2} > 0$ so β_2 and α_2 must have the same sign, in which case $a = (\beta_2/\alpha_2)^{1/(n-2)}$. We have thus established the following theorem.

Theorem 2. There are exactly three affine equivalence classes of polynomials of even degree $n \geq 4$. The polynomials $x^n + x^2$, x^n, and $x^n - x^2$ are representatives of the three classes. For polynomials of odd degree $n \geq 4$, there are but two affine equivalence classes, represented by $x^n + x^2$ and x^n.

4. PROPORTIONALITY OF ARCS AND CHORDS OF A PARABOLA. Turning now to the proportionality results in [**3**], consider the parabola $y = x^2$. For given positive numbers a and m, define an affine transformation T by

$$T(x, y) = \left(ax + m, 2amx + a^2y + m^2\right).$$

According to properties (iii) and (iv), T maps the parabola onto itself and transforms areas by a factor of a^3. Also, T maps the origin $(0, 0)$ to the point (m, m^2), where the line $y = mx$ intersects the parabola.

Let $p_0 = (0, 0)$ and define $p_n = T(p_{n-1})$ for each natural number n. Then T maps the parabolic arc and the chord connecting the points p_{n-1} and p_n to the arc and the chord connecting the points p_n and p_{n+1}. It follows that the area of the region enclosed by the second arc and chord is a^3 times the area of the region enclosed by the first arc and chord. Thus, the number a^3 here takes the place of the proportionality constant k in [**3**].

Before looking at the lengths of the parabolic arcs and chords just mentioned, notice that the x-coordinate of the point p_n, denoted by a_n in keeping with [**3**], is equal to

$$
\begin{aligned}
a_n &= a(\cdots(a(a(a\cdot 0+m)+m)+m)\cdots)+m\\
&= m(a^{n-1}+\cdots+a+1)\\
&= \begin{cases} mn, & \text{if } a=1,\\ m\left(\dfrac{1-a^n}{1-a}\right), & \text{if } a\neq 1.\end{cases}
\end{aligned}
$$

This agrees with the result in [**3**], keeping in mind that $k = a^3$ now. Also, for $0 < a < 1$, we see that $\lim_{n\to\infty} a_n = m(1-a)^{-1}$, while for $a \geq 1$, $a_n \to \infty$ as $n \to \infty$.

As in [**3**], let d_n denote the length of the chord connecting p_n to p_{n+1}, and let L_n denote the length of the parabolic arc connecting these same two points. We handle separately the two cases $a \geq 1$ and $0 < a < 1$.

If $a \geq 1$, the points p_n go off to infinity along the parabola, so the successive chords and arcs become increasingly vertical as $n \to \infty$. By property (iv), the affine transformation T maps vertical segments to vertical segments and changes the lengths of such segments by the factor a^2. It follows, therefore, that the ratios L_{n+1}/L_n and d_{n+1}/d_n both tend to the limiting value a^2 as $n \to \infty$. This is the result we found in [**3**].

The case where $0 < a < 1$ requires only a bit more effort. In this case, the points p_n converge to the point $p_* = (m(1-a)^{-1}, m^2(1-a)^{-2})$, which is a fixed point for T. Indeed, $T(p_*) = T(\lim p_n) = \lim T(p_n) = \lim p_{n+1} = p_*$. It follows that the line $\mathcal{L}$ tangent to the parabola at p_* is mapped onto itself by T. The equation for $\mathcal{L}$ is

$$
y = \left(\frac{2m}{1-a}\right)x - \frac{m^2}{(1-a)^2},
$$

and a not-too-tedious calculation shows that T multiplies the lengths of segments of $\mathcal{L}$ by the factor a.

As $n \to \infty$, both the arc and the chord connecting p_n to p_{n+1} are approximated by segments of the line $\mathcal{L}$, so

$$
\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \lim_{n\to\infty} \frac{d_{n+1}}{d_n} = a.
$$

This agrees with [**3**] and completes the main result there.

REFERENCES

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Villanova University, Villanova, PA 19085-1699
timothy.feeman@villanova.edu
osvaldo.marrero@villanova.edu