

Equal Area World Maps: A Case Study*

Timothy G. Feeman
Dept. of Mathematical Sciences
Villanova University
Villanova, PA 19085
USA
tfeeman@email.vill.edu

Introduction

If one wishes to produce a world map that displays area-based data, such as the extent of rain forests, the range of butterfly migrations, or the access of people in various regions to medical facilities, then it is often appropriate to use a base map that shows the areas of all regions of the earth's surface in their correct proportions. Such a map is called an *equal area* or *equivalent* map by cartographers.

A few common equal area maps are Lambert's equal area cylindrical projection, the Gall-Peters equal area cylindrical projection, and Albers' equal area conical projection, presented in 1805 and still used by the United States Geological Survey. In this article, we will examine the step-by-step construction of another equal area map known as the Mollweide projection. First announced in 1805 by Karl Brandan Mollweide (1774–1825) and commonly used in atlases today, this map projection represents the whole world in an ellipse whose axes are in a 2:1 ratio.

Aesthetically, the elliptical shape of Mollweide's map reflects the essentially round character of the earth better than rectangular maps. To add to

*This research was partially supported by the grant NSF-DUE-95-52464

its visual appeal, the Mollweide map incorporates the fact that, if we were to look at the earth from deep space, we would see only one hemisphere which would appear circular to us. Thus, on this map, the hemisphere facing us as we look at the globe is depicted as a central circle, with the diameter of the circle equal to the vertical axis of the overall ellipse. The “dark side” of the earth is split in two with one piece shown on either side of the central circle.

The mathematics involved in the construction below requires mainly high school algebra and trigonometry with only a bit of Calculus (which can be plausibly avoided if one so desires). All of this material, except for the Calculus, was included in a first-year liberal arts math/geography course taught at Villanova University. (See <http://www.villanova.edu/~tfeeman/> for more details on the course.) The map at the end of the article was produced by Russell Wolff, a student in the course.

1 Constructing the Mollweide map

For simplicity, let's use as our model for the earth a spherical globe of radius 1. Also, the map construction given here will have the point at 0° latitude and 0° longitude as its center, though any point can serve as the center with the appropriate changes in the calculations. (The modifications required are minor if the central point is on the equator.)

As mentioned above, the entire globe will be represented within an ellipse. The horizontal axis of the ellipse represents the equator while the vertical axis depicts the Prime Meridian. To complete the overall structure of the map, the parallels (circles on the globe of constant latitude) will be drawn as horizontal lines on the map (as they would appear to be if we could see them from space), while the two meridians at v° East and West will together form an ellipse whose vertical axis coincides with the vertical axis of the overall ellipse. With this basic setup, let's turn to the problem of preserving areas.

1.1 Computing areas

The total surface area of the globe of radius 1 is, of course, 4π . More generally, the area of the portion of the sphere lying between the equator and the latitude parallel at u° North is equal to $2\pi \sin(u^\circ)$. This can be proved using Calculus by taking the equation $z = \sqrt{1 - x^2 - y^2}$ for the upper hemisphere

and computing the surface area element

$$dA = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dx dy = \frac{1}{\sqrt{1-x^2-y^2}} dx dy.$$

The region of integration is a ring with inner radius $r = \cos(u^\circ)$ and outer radius $r = 1$. Switching to cylindrical coordinates and integrating the area element over this region yields

$$\text{area} = \int_{\theta=0}^{2\pi} \int_{r=\cos(u^\circ)}^1 \frac{r}{\sqrt{1-r^2}} dr d\theta = 2\pi \sin(u^\circ).$$

Notice that the area of a hemisphere, 2π , is obtained by taking $u = 90$.

To plausibly justify this formula without using Calculus, start with the fact that most college students remember having learned that the surface area of a sphere of radius 1 is 4π . Thus the area of a hemisphere is 2π . Looking at a sphere convinces us that the area of a band bounded by the equator and the parallel at u° is concentrated nearer to the equator. That is, the vertical component of the latitude angle, $\sin(u^\circ)$, is what counts when measuring the area of the band.

The central circle on the Mollweide map is to represent a hemisphere, so its area must be 2π , as we have just computed. Thus, the radius of the central circle must be $\sqrt{2}$. We will take this as the length of the semi-minor (vertical) axis of the overall ellipse as well.

The area of an ellipse with semi-axes of lengths a and b is equal to πab . This can be proved using Calculus by taking the equation $x^2/a^2 + y^2/b^2 = 1$ for the ellipse, solving for y , and integrating. By symmetry, it is enough to compute the area in the first quadrant. Thus,

$$\text{area of ellipse} = \frac{4b}{a} \int_{x=0}^a \sqrt{a^2 - x^2} dx = \pi ab.$$

To avoid Calculus, one can make a plausibility argument for this formula by comparing it to the formula for the area of a circle, where $a = b$.

For the Mollweide map, we have already determined that $b = \sqrt{2}$ and that the total area is 4π . Thus, the semi-major (horizontal) axis of the overall ellipse must have length $a = (4\pi)/(\sqrt{2}\pi) = 2\sqrt{2}$. That is, the overall ellipse is twice as long as it is high.

1.2 Placing the parallels

To determine the placements of the parallels, recall from the preceding paragraphs that the area of the portion of the sphere lying between the equator and the latitude parallel at u° is equal to $2\pi \sin(u^\circ)$. Half of this, or $\pi \sin(u^\circ)$, will lie in the hemisphere shown in the central circle of the map.

Now suppose that we place the image of the parallel for u° (north) at height h above the equator on the map. Let t° be the angle between the equator and the radius of the central circle corresponding to the parallel, as shown in Figure 1. So $h = \sqrt{2} \sin(t^\circ)$. The area of the portion of the central

Figure 1: The Mollweide Map – placing the parallels

circle which lies between the equator and this horizontal line (the image of the parallel) is made up of two circular sectors, each made by an angle of t° , and two right triangles, each with adjacent sides of lengths h and $\sqrt{2} \cos(t^\circ)$. Since the whole circle has area 2π , each sector has an area of

$$\text{sector area} = 2\pi \frac{t}{360} = \frac{\pi t}{180}.$$

Each of the right triangles has area

$$\text{triangle area} = \frac{1}{2} h \sqrt{2} \cos(t^\circ) = \sin(t^\circ) \cos(t^\circ) = \frac{1}{2} \sin(2t^\circ),$$

since $h = \sqrt{2} \sin(t^\circ)$. For our map to preserve areas, the sum of the areas of the two sectors and the two right triangles must be equal to $\pi \sin(u^\circ)$. Thus, we have the following equation to solve for t in terms of u .

$$\pi \sin(u^\circ) = \frac{\pi t}{90} + \sin(2t^\circ). \quad (1)$$

To get a general formula for t in terms of u from this equation turns out to be rather unwieldy. For example, Maple was unable to produce a general solution. Instead, we can use the equation to estimate the values of t that correspond to the latitudes u that we actually wish to show on our map. For example, substituting $u = 30$ into the left hand side of equation 1 and solving for t gives us $t \approx 23.826771$ degrees; so, the corresponding height for the image of the parallel at 30° is $h = \sqrt{2} \sin(23.826771^\circ) \approx .571304$. Table 1, generated using Maple, shows the approximate values of t and h for latitudes in 10 degree increments.

$\pi \sin(u^\circ) = \frac{\pi t}{90} + \sin(2t^\circ)$		
u°	t°	$h = \sqrt{2} \sin(t^\circ)$
0	0	0
10	7.86335	.19348
20	15.78419	.38469
30	23.82677	.57130
40	32.07120	.75091
50	40.62893	.92088
60	49.67500	1.07818
70	59.53172	1.21892
80	70.97783	1.33699
90	90	$\sqrt{2}$

Table 1: Placement of the parallels

For the parallels to fit exactly into the overall ellipse of the map, we need to know how long to draw each one. To compute this, recall that the rectangular coordinates (x, y) of each point on an ellipse with semi-axes of lengths $2\sqrt{2}$ and $\sqrt{2}$ in the horizontal and vertical directions, respectively, are related by the equation

$$\frac{x^2}{8} + \frac{y^2}{2} = 1. \quad (2)$$

For the parallel at height h we substitute $y = h$ into this equation to get

$$\frac{x^2}{8} + \frac{h^2}{2} = 1. \quad (3)$$

Solving this for x in terms of h yields

$$x = \pm 2\sqrt{2 - h^2}, \quad (4)$$

so the length of the parallel at height h is $4\sqrt{2 - h^2}$.

1.3 Placing the meridians

Having taken care of the placement of the parallels, we now turn to the placement of the meridians, which, recall, are to form ellipses when taken in pairs. We want the vertical axis of the ellipse formed by the meridians at v° East and West to be the same as the vertical axis of the central circle. So, its length is $2\sqrt{2}$. This means that the length of the semi-axis in the vertical direction is to be $\sqrt{2}$ no matter what v is. The length of the axis in the horizontal direction, on the other hand, will depend on v . Because our map is supposed to show correct areas, the key to determining the right length for the horizontal axis is to look at two areas, one on the sphere and one on the map, and make sure that they are equal. First, on the map, suppose for the moment that we denote by $2a$ the length of the horizontal axis of the ellipse formed by the meridians at v° East and West (so a is the semi-axis in the horizontal direction). Since the vertical semi-axis is $\sqrt{2}$ and since the area of an ellipse is π times the product of the lengths of the two semi-axes, we get that the area of the ellipse on the map is

$$\text{area on map} = \pi\sqrt{2}a. \quad (5)$$

Now look at the corresponding portion of the sphere; that is, consider the portion of the sphere bounded by the meridians at v° East and West. This represents an angle of $2v$ degrees, out of the 360 degrees possible. So, the area of this region (technically called a *lune*) is the fraction $2v/360$ of the total surface area of the sphere (which has radius 1). Thus,

$$\text{area on sphere} = \frac{2v}{360} \cdot 4\pi = \frac{\pi v}{45}. \quad (6)$$

Setting these two areas equal, we get

$$\pi\sqrt{2}a = \frac{\pi v}{45}. \quad (7)$$

Solving this for a in terms of v , we get

$$a = \frac{v}{45\sqrt{2}} = \frac{v\sqrt{2}}{90}. \quad (8)$$

The ellipse we want on the map, then, has semi-axes of lengths $v\sqrt{2}/90$ in the horizontal direction and $\sqrt{2}$ in the vertical direction. The equation of the ellipse is therefore given by

$$\frac{x^2}{\left(\frac{v\sqrt{2}}{90}\right)^2} + \frac{y^2}{(\sqrt{2})^2} = 1, \quad (9)$$

or, in simplified form,

$$\frac{4050 \cdot x^2}{v^2} + \frac{y^2}{2} = 1. \quad (10)$$

This ellipse can be plotted, for specified values of v , using Maple or some other computer graphics package. Note that, if we take $v = 90$, we get $x^2/2 + y^2/2 = 1$ or, more simply, $x^2 + y^2 = 2$. This is the equation of the central circle, just as it should be. Also, the equation of the overall ellipse for the Mollweide map, $x^2/8 + y^2/2 = 1$, is obtained by taking $v = 180$.

Below is a picture of the grid, with latitudes in 10-degree increments and longitudes in 20-degree increments, obtained from the equations above. Following that (thanks again to Russell Wolff) is a Mollweide map showing the continental boundaries for the southern hemisphere. This map is centered on the meridian at 60° East longitude.

Figure 2: Graticule for the Mollweide map

Figure 3: A Mollweide map of the southern hemisphere

References

BUGAYEVSKIY, LEV M., and JOHN P. SNYDER. 1995. *Map Projections: A Reference Manual*. London: Taylor and Francis.

DEETZ, CHARLES H., and OSCAR S. ADAMS. 1969. *Elements of Map Projection*. New York: Greenwood Press.

SNYDER, John P.. 1993. *Flattening the Earth: Two Thousand Years of Map Projections*. Chicago: University of Chicago Press.