A Correction to
Affine transformations, polynomials, and proportionality
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We would like to thank several careful readers of our article [1] for pointing out to us that Theorem 2 therein is incorrect as stated. The other results in the paper, concerning a variety of proportionality problems for parabolas and cubic curves, are not affected by this error. The correct statement of the theorem should be as follows.

Theorem 2’. Let \( n \geq 4 \) and let \( p(x) = \sum_{j=0}^{n} \alpha_j x^j \) and \( q(x) = \sum_{j=0}^{n} \beta_j x^j \) be polynomials such that \( \alpha_n = \beta_n = 1 \), \( \alpha_{n-1} = \beta_{n-1} = 0 \), and \( \alpha_0 = \beta_0 = 0 \). Then \( p \) and \( q \) are affinely equivalent if, and only if, there exists a non-zero number \( a \) such that \( \beta_j = a^{n-j} \alpha_j \) for all \( j \neq 1 \). In this case, the affine transformation \( T(x, y) = (ax, (a\beta_1 - a^n \alpha_1)x + a^ny) \) implements the equivalence.

Proof. As discussed in [1], the restrictions on the coefficients of \( p \) and \( q \) involve no loss of generality as every polynomial of degree \( n \) is affinely equivalent to one such as this.

Assume that \( p \) and \( q \) are affinely equivalent, implemented by the affine transformation \( T(x, y) = (ax+by+e, cx+dy+f) \) with \( ad-bc \neq 0 \). Thus, \( q(ax+bp(x)+e) = cx+dp(x)+f \) for all \( x \). Comparisons of the coefficients of the polynomials \( q(ax+bp(x)+e) \) and \( cx+dp(x)+f \) show that \( b = e = f = 0 \) and that the non-zero number \( a \) satisfies \( \beta_j = a^{n-j} \alpha_j \) for all \( j \neq 1 \). Moreover, \( c = a\beta_1 - a^n \alpha_1 \) and \( d = a^n \).

Conversely, if there is a non-zero number \( a \) for which \( \beta_j = a^{n-j} \alpha_j \) whenever \( j \neq 1 \), then it is straightforward to verify that the affine transformation \( T(x, y) = (ax, (a\beta_1 - a^n \alpha_1)x + a^ny) \) satisfies the condition \( T(x, p(x)) = (ax, q(ax)) \) for all \( x \). That is, \( T \) implements an affine equivalence between \( p \) and \( q \).

For \( n = 4 \), this result implies that there are exactly three affine equivalence classes of quartic polynomials, represented by \( x^4 \), \( x^4 + x^2 \), and \( x^4 - x^2 \). For \( n \geq 5 \), however, there are infinitely many affine equivalence classes. For instance, the quintics \( x^5 + \alpha x^3 + x^2 \) and \( x^5 + \beta x^3 + x^2 \) are affinely equivalent if, and only if, \( \alpha = \beta \).

REFERENCE


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